Affine planes, coordinatization and cartesian groups

Definition of an affine plane. An affine plane consists of a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ such that each line is a subset of $\mathcal{P}$ and the following holds:

(A1) $\forall x, y \in \mathcal{P}$: $x \neq y \Rightarrow \exists! \ell \in \mathcal{L}$ such that $x, y \in \ell$;
(A2) $\forall x \in \mathcal{P}, s \in \mathcal{L} : x \notin s \Rightarrow \exists! t \in \mathcal{L}$ such that $x \in t$; and
(A3) $\exists x_i \in \mathcal{P}$, $1 \leq i \leq 3$, such that $\{x_1, x_2, x_3\} \subseteq \ell$ for no line ℓ.

Property (A3) may be expressed by saying that there exist three points that are not collinear.

Remarks:

(a) In the definition none of $\exists!$ may be replaced by $\exists$. To get a counterexample for (A1) double a point; for (A2) consider affine lines in a 3-dimensional vector space.
(b) Suppose that $r, s, t \in \mathcal{L}$ are such that $r \cap s = \emptyset = s \cap t$ and $r \neq t$. There has to be $r \cap t = \emptyset$ since $x \in r \cap t$ contradicts condition (A2).
(c) Write $s \parallel t$ if $s, t \in \mathcal{L}$ are such that either $s = t$ or $s \cap t = \emptyset$. From (b) it follows immediately that $\parallel$ is an equivalence on $\mathcal{L}$.
(d) *Pencil* is the traditional word used for a class of the equivalence $\parallel$ (in Czech 'svazek').

Following facts are immediate to establish:

(i) Each line has at least two elements;
(ii) if $s, t \in \mathcal{L}$ and $s \cap t \neq \emptyset$, then $\mathcal{P} \neq s \cup t$;
(iii) there are at least 3 classes of parallelism (i.e., at least three pencils);
(iv) all lines are of the same size (cardinality). In finite case the number of points upon a line is called the *order* of the plane.
(v) In an affine plane of finite order n there are exactly $n+1$ parallel classes, each consisting of n lines. The number of points is equal to n^2.
(vi) From an affine plane a projective plane may be constructed by forming 'a point at infinity' for each pencil, and 'a line at infinity' that connects all points at infinity.
(vii) Removing a line from a projective plane yields an affine plane.

The equivalence $\parallel$ satisfies this property:

(A4) For any $\ell \in \mathcal{L}$ the class (pencil) $[\ell]_{\parallel}$ partitions $\mathcal{P}$.

Let us now suppose that upon $\mathcal{L}$ there is given an equivalence $\parallel$ that satisfies (A4), and we look for conditions under which this equivalence really is the equivalence of the parallelism within an affine plane.

Let $\parallel$ be an equivalence on $\mathcal{L}$. If $\parallel$ satisfies (A4), then $(\mathcal{P}, \mathcal{L})$ is an affine plane with parallelism $\parallel$ if and only if (A1) and (A3-A5) are satisfied.

(A5) if $s, t \in \mathcal{L}$ and $s \parallel t$ does not hold, then $\exists!\ x \in s \cap t$.

Proof. Extend the set of points by regarding each class of $\parallel$ as a point in which the lines of the class meet, and extend the set of lines by adding a line consisting of all these points. Then (A1) and (A3)–(A5) imply that the extended structure fulfils axioms of the projective plane. By removing the added line at infinity we obtain the desired result. $\square$

Axiomatics of projective planes immediately implies that in conditions (A1) and (A5) one of $\exists!$ (but not both) may be replaced by $\exists$.

Conditions for the finite case. Suppose that $\mathcal{P}$ is finite. Let $\parallel$ be an equivalence upon $\mathcal{L}$. Consider the following condition (which may be regarded as a stronger version of (A4)).

(A6) If $\ell \in \mathcal{L}$, then $[\ell]_\parallel$ partitions $\mathcal{P}$ into $n \geq 2$ classes, each of which is of size n, and there are $n+1$ classes (pencils) of $\parallel$.

If (A6) is fulfilled, then $(\mathcal{P}, \mathcal{L})$ is an affine plane provided (A7) or (A8) is true.

(A7) If $x, y \in \mathcal{P}$, $x \neq y$, then there exists at most one $\ell \in \mathcal{L}$ such that $x, y \in \ell$;
(A8) if $s, t \in \mathcal{L}$ are such that $s \parallel t$ does not hold, then $s \cap t \neq \emptyset$.

Proof. Assume first that (A6) and (A7) hold. Any two points determine at most one line. There are n^2 points and $\binom{n^2}{2}$ pairs of points. Every line contains $\binom{n}{2}$ pairs of points, there are n lines in a pencil, and there are $n+1$ pencils. Since $(n+1)n\binom{n}{2} = \binom{n^2}{2}$, any pair of points is contained in a line. Therefore any two points occur in exactly one line. What remains is to verify (A5). Suppose that $s \parallel t$ does not hold. There are n lines parallel to s. None of them may share with t two points or more. Since t has exactly n points, t intersects each of them in exactly one point.

Let us show that (A6) and (A8) imply (A7). Proceed by contradiction, assuming that $x, y \in s \cap t$, where $x \neq y$ and $s \neq t$ are lines. By this assumption $s \parallel t$ does not hold. Let $t = t_1, \dots, t_n$ be all the lines parallel to t. Since s intersects each of them and since s consists of n points, there has to be $|s \cap t_i| = 1$, for every $i \in \{1, \dots, n\}$. That contradicts the assumption. $\square$

Collineations and the line at infinity. A *collineation* ψ of an affine plane is a permutation of points that maps a line upon a line. This is the same as saying that $x \in \ell$ if and only if $\psi(x) \in \psi(\ell)$.

A collineation of an affine plane thus respects parallelism. Therefore it permutes pencils. This may be also expressed by saying that it permutes points at infinity. In this way each collineation of an affine planes extends uniquely to a collineation of its projective completion. On the other hand, a collineation of projective plane that fixes a line pointwise yields a collineation of the affine plane that is obtained by removal of the line.

Denote the line at infinity by ℓ_∞. Let us characterize collineations α of the affine plane that are induced by a perspectivity for which ℓ_∞ is the axis. What we shall do is to take the description of perspectivity in projective plane, and explain what it means in this special case. Note first that α *maps each line upon a parallel line.*

Denote by c the center of the perspectivity and choose an affine point $x \neq c$. Suppose that α is nontrivial. Therefore $\alpha(x) \neq x$. Denote by ℓ the affine line connecting x and $\alpha(x)$. Thus $\alpha(\ell) = \ell$ and ℓ passes through c. The goal is to determine $\alpha(y)$, where y is another affine point, $y \notin \ell$. The affine line connecting x and y will be denoted by m. We know that $m' = \alpha(m) \parallel m$ and $\alpha(x) \in m'$. This determines m' completely.

Now, $\alpha(y)$ is the intersection of m' and an affine line ℓ' such that $y \in \ell'$ and either $\ell' \parallel \ell$—if α is induced by *elation*—or $c \in \ell'$—if α is induced by homology.

Introducing binary operations. Let $(X, +, 0)$ be a group (not necessarilly commutative, despite the notation). Suppose that upon X there is also defined a binary operation $\cdot$ such that $0x = x0 = 0$ for all $x \in X$. Write $X^* = X \setminus \{0\}$. Put $\mathcal{P} = X \times X$. For $a \in X$ let $\ell_a = \{(x, y) \in \mathcal{P};\ x = a\}$. If $a, b \in X$, set $\ell_{a,b} = \{(x, y) \in \mathcal{P};\ y = xa + b\}$. Let $\ell_{\infty,a}$ be an alternative notation for ℓ_a, and set $X_\infty = X \cup \{\infty\}$. Set $\mathcal{L} = \{\ell_{a,b};\ (a, b) \in X_\infty \times X\}$ and assume that each 'line' possesses a unique name, i.e., that $\ell_{a,b} = \ell_{a',b'}$ implies $a = a'$ and $b = b'$, whenever $a, a' \in X_\infty$ and $b, b' \in X$.

What can be said about $\cdot$ if $(\mathcal{P}, \mathcal{L})$ is an affine plane? Let us first observe that *left translations* $L_a \colon x \mapsto ax$ and *right translations* $R_a \colon x \mapsto xa$ have to permute X

for each $a \in X^* = X \setminus \{0\}$. (Note that $L_0 = R_0$ is a constant mapping that sends each element of X to 0.)

R_a inj.: Suppose that $c_1 a = c_2 a = d$. Then $(c_i, d) \in \ell_{a,0} \cap \ell_{0,d}$. Hence $c_1 = c_2$.

R_a surj.: For $d \in X$ consider the intersection (x, y) of $\ell_{0,d}$ and $\ell_{a,0}$. This means that $d = y = xa$. Hence $d \in \mathrm{Im}(R_a)$.

L_a inj.: Suppose that $ac_1 = ac_2 = d$. Then both (a, d) and $(0, 0)$ belong to both $\ell_{c_1,0}$ and $\ell_{c_2,0}$. The lines thus agree, and hence $c_1 = c_2$.

L_a surj.: For $d \in X$ consider the line that connects $(0, 0)$ and (a, d). It is not of the form ℓ_c since $a \neq 0$. Hence it is one of lines $\ell_{c,b}$, $c \in X$. Since $0 = 0c + b$, there has to be $b = 0$. Therefore $d = ca$.

As a consequence:

$$ab = 0 \;\Leftrightarrow\; 0 \in \{a, b\}.$$

Translations R_a and L_a thus permute X^*, for every $a \in X^*$. This is the definition of a *quasigroup*. The fact that $(X^*, \cdot)$ is a quasigroup is thus a necessary condition for $(\mathcal{P}, \mathcal{L})$ to be an affine plane. To get a necessary condition let us first make this observation:

If a is fixed and $\alpha = (u, v) \in \mathcal{P}$, then there exists a unique line $\ell_{a,b}$ with $\alpha \in \ell_{a,b}$. Indeed, if $a = \infty$, then $b = u$. If $a \in X$, then b is uniquely determined by $v = au+b$.

This means that (A4) is satisfied if $\parallel$ is an equivalence of $\mathcal{L}$ such that $\ell_{a,b} \parallel \ell_{a',b'} \;\Leftrightarrow\; a = a'$. If this is assumed, then, as we shall observe, (A1) and (A5) may be reduced to

(C1) If $u, u', v, v' \in X$ are such that $u \neq u'$, then there exists a unique $x \in X$ such that $-ux + v = -u'x + v'$; and

(C2) if $u, u', v, v' \in X$ are such that $u \neq u'$, then there exists a unique $x \in X$ such that $xu + v = xu' + v'$.

Indeed, if $\alpha = (u, v)$ and $\alpha' = (u', v')$ are two distinct points, then ℓ_u is the unique line containing both α and α' if $u = u'$. Assume $u \neq u'$. The existence of unique $(a, b) \in X \times X$ with $ua + b = v$ and $u'a + b = v'$ is equivalent to the existence of unique a such that $-ua + v = -u'a + v'$, and that is how (A1) and (C1) are connected.

To connect (A5) and (C2) consider $a, a', b, b' \in X$. Lines $\ell_{a'}$ and $\ell_{a,b}$ intersect at $(a', a'a + b)$. Assume $a' \neq a$. Lines $\ell_{a,b}$ and $\ell_{a',b'}$ intersect at (x, y) if and only if $xa + b = xa' + b'$.

If $(Q, +, 0)$ is a group, then (C1) and (C2) may be simplified. The advantage of the unsimplified form rests in the fact that (C1) and (C2) imply (A1) and (A5) also in the case when $(Q, +, 0)$ is a *loop*, i.e., a quasigroup in which 0 is the neutral element. (In the loop case $-a + b$ is the solution of $a + x = b$, while $a - b$ is the solution to $x + b = a$.)

We have verified that *if (C1) and (C2) hold, then $(\mathcal{P}, \mathcal{L})$ is an affine plane in which $\ell_{a,b} \parallel \ell_{a',b'} \Leftrightarrow a = a'$.*

If X is finite, then $\mathcal{L}$ satisfies (A6). This means that in finite case it is possible to relax (C1) and (C2) by using (A7)/(A8) in place of (A1) and (A5). The details will be skipped since the focus here is upon the situation when $(X, +, 0)$ is a group. In such a case (C1) and (C2) may be expressed, in the respective order, as follows:

(C3) If $u, u', v \in X$, $u \neq u'$, then $\exists!\ x \in X$ such that $u'x = v + ux$; and

(C4) if $u, u', v \in X$, $u \neq u'$, then $\exists!\ x \in X$ such that $xu' = xu + v$.

Theorem. *Suppose that $(X, +, 0)$ is a nontrivial group and that $\cdot$ is a binary operation upon X such that (C3) and (C4) hold, and that $x \cdot 0 = 0 = 0 \cdot x$ for each $x \in X$. Then $(X^*, \cdot)$ is a quasigroup, where $X^* = X \setminus \{0\}$, and $(\mathcal{P}, \mathcal{L})$ is an affine plane, where $\mathcal{P} = X \times X$ and $\mathcal{L} = \{\ell_a, \ell_{a,b};\ a, b \in X\}$, $\ell_a = \{(a, x);\ x \in X\}$ and $\ell_{a,b} = \{(x, xa + b),\ x \in X\}$.*

The mapping $\alpha_c\colon (x,y)\mapsto(x,y+c)$ is a collineation of $(\mathcal{P},\mathcal{L})$ for each $c\in X$. If $\ell\in\mathcal{L}$, then $\alpha_c(\ell)\parallel\ell$. If $\ell=\ell_a$, $a\in X$, then $\alpha_c(\ell)=\ell$.

Put $(a)=\{\ell_{a,b};\ b\in X\}$ for each $a\in X$, and set $(\infty)=\{\ell_a;\ a\in X\}$. Consider the projective plane obtained from $(\mathcal{P},\mathcal{L})$ by adding the line at infinity $\ell_\infty=\{(a);\ a\in X\}\cup\{(\infty)\}$. This projective plane is $((\infty),\ell_\infty)$-transitive.

Proof. By the preceding claim, $(\mathcal{P},\mathcal{L})$ is an affine plane if $(X^*,\cdot)$ is a quasigroup. To see that the latter follows from (C3) and (C4) assume that $u=0$, $u'\neq 0$ and $v\neq 0$. Each of $u'x=v$ and $xu'=v$ has exactly one solution, and this solution is from X^*. This implies that $(X^*,\cdot)$ is a quasigroup.

The mapping α_c sends (a,y) to $(a,y+c)$. Hence it maps ℓ_a upon itself. Furthermore, $(x,xa+b)$ is sent upon $(x,xa+b+c)$. This means that $\alpha_c(\ell_{a,b})=\ell_{a,b+c}$.

Nothing else needs to be proved. □

A system $(X,+,\cdot,0,1)$ is called a *cartesian group* if $(X,+,0)$ is a group, (C3) and (C4) hold, $0\neq 1$, and each $x\in X$ satisfies both $x0=0=0x$ and $x1=x=1x$. By the theorem, each cartesian group yields a $((\infty),\ell_\infty)$-transitive projective plane. Below we shall observe that if a projective plane is (c,a)-transitive, then it may be described by a cartesian group. The assumption concerning 1 is equivalent to saying that $(X^*,\cdot,1)$ is a loop. As we have seen, the assumption that $\cdot$ posseses a unit element is not needed to get the $((\infty),\ell_\infty)$-transitivity. However, the fact that it may be made makes work with the algebraic system easier. Further remarks:

(i) As follows from the comment after introducing (A5), in (C3) and (C4) one of $\exists!$ may be replaced by $\exists$.

(ii) There are definitions of cartesian groups in which (C3) and (C4) swap the order of arguments. For example the equation $u'x=v+ux$ may appear as $u'x=ux+v$. This may be harmonized by using the mirror operation (i.e., the opposite group or opposite loop).

(iii) Sometimes the system described in the theorem is called a *weak cartesian group*. A weak cartesian group is thus a cartesian group without the unit.

In finite case (A7)/(A8) imply (A1) and (A5) if (A6) is fulfilled. This has several consequences for finite weak cartesian groups:

Proposition. *Assume that X is finite, $(X,+,0)$ is a nontrivial group, $(X^*,\cdot)$ is a quasigroup, and $x\cdot 0=0=0\cdot x$ for all $x\in X$. Then $(X,+,\cdot,0)$ is a weak cartesian group if and only if at least one of (C3) and (C4) holds. Furthermore, (C3) holds if (a) is true or (b) is true. Similarly for (C4).*

(a) *The equation $u'x=v+ux$ has at least one solution x whenever $u,u',v\in X$ and $u\neq u'$; and*

(b) *the equation $u'x=v+ux$ has at most one solution x whenever $u,u',v\in X$ and $u\neq u'$.*

Proof. Expressing (A7) and (A8) in terms of (C3) and (C4) shows that the only step needed to do is to explain why (a) $\Leftrightarrow$ (b). This is clear if $u'=0$ or $u=0$ since $(X^*,\cdot)$ is quasigroup.

Put $T=\{(u',u,v);\ u,u'\in X^*,\ u\neq u'$ and $v\in X\}$. Set $F=\{(u',u,v,x);\ (u',u,v)\in T,\ x\in X$ and $u'x=v+ux\}$. We have $|F|=n(n-1)(n-2)$ since for each $(u',u,x)\in T$ there exists exactly one $v\in X$ such that $(u',u,v,x)\in F$.

Our aim now is to count $|F|$ by different means. For $u\in X$ let L_u be the left translation $x\mapsto ux$. If $(u',u,v)\in T$, then $u'x=v+ux$ if and only if $x=\tau_{u',u,v}(x)$, where $\tau_{u',u,v}(x)=L_{u'}^{-1}(v+L_u(x))$. The size of F hence coincides with the aggregate number of points fixed by permutations τ_α, $\alpha\in T$. Since $|T|=|F|$, the average number of points fixed by τ_α is equal to 1. Hence if there exists a mapping τ_α that fixes no point, or fixes two or more points, then both these alternatives take place.

If τ_α fixes no point, then (a) is violated. If τ_α fixes more than one point, then (b) is violated. □

Dualization. As follows from (C3) and (C4), the notion of (weak) cartesian group is self-dual. This means that if $\circ$ and $\oplus$ are binary operations upon X such that $x \circ y = yx$, $x \oplus y = y + x$, and $(X, +, \cdot, 0)$ is a weak cartesian group, then $(X, \oplus, \circ, 0)$ is a weak cartesian group as well. The aim now is to show that $(X, \oplus, 0, \circ)$ may be used to coordinatize the dual projective plane.

Let $(X, +, \cdot, 0)$ be a weak cartesian group. As proved above, the induced projective plane is $((\infty), \ell_\infty)$-transitive. The line ℓ_∞ seen as a dual point will be denoted by $[\infty]$. The point (∞) seen as a dual line will be denoted by $\tilde{\ell}_\infty$. The dual projective plane is $([\infty], \tilde{\ell}_\infty)$-transitive. The dual points at infinity are the lines that pass through $\tilde{\ell}_\infty = (\infty)$, i.e., the lines ℓ_a, $a \in X_\infty$. Write ℓ_a as $[a]$ if ℓ_a is regarded as a dual point. The dual affine points are thus all of the lines $\ell_{a,b}$, where $a, b \in X$. Denote $\ell_{a,b}$ as $[a, -b]$ when regarded as a dual point. Thus $[a, b] = \ell_{a,-b}$.

Points of the projective plane are the dual lines. Each dual line may be identified with the set of lines (i.e., dual points) that are concurrent to the given point (i.e., incident to the given dual line). Let us determine for each dual line (i.e., for each point) the dual points incident to the dual line (i.e., the lines concurrent to the given point).

The lines passing through (∞) are the lines ℓ_a, $a \in X_\infty$. Hence $(\infty) = \tilde{\ell}_\infty$ consists of dual points $[a] = \ell_a$, $a \in X_\infty$.

The lines passing through (a), $a \in X$, are the lines $\ell_{a,x} = [a, -x]$, $x \in X$. The dual affine plane thus contains lines $\tilde{\ell}_a = \{[a, x];\ x \in X\}$.

Consider the lines passing through $(u, -v)$, where $u, v \in X$. We have $(u, -v) \in \ell_{a,-b} = [a, b]$ if $ua - b = -v$. That is equivalent to $-b = -ua - v$ and to $v + ua = b$. This means that $(u, -v)$ belongs to those lines that may be expressed as dual points $[x, v + ux]$. Hence $\tilde{\ell}_{(u,v)} = \{[x, v + ux];\ x \in X\}$ is a dual affine line. We may state:

Consider an affine plane coordinatized by a weak cartesian group $(X, +, \cdot, 0)$. The dual of the induced projective plane from which the dual line $\tilde{\ell}_\infty = (\infty)$ is removed yields an affine plane that may be coordinatized by the opposite weak cartesian group. Its points are $[u, v] = \ell_{u,-v}$, where $u, v \in X$. The lines are $\tilde{\ell}_a = (a) = \{[a, x];$ $x \in X\}$ and $\tilde{\ell}_{a,b} = \{[x, b + ax];\ x \in X\} = (a, -b)$, for all $a, b \in X$.

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The aim of coordinatization. As suggested before, a (c, a)-transitive projective plane may always be described by a cartesian group. This will be proved within a broader context of introducing coordinates to projective and affine planes. This process is called *coordinatization.* Its aim is to express geometric notions in algebraic terms. The first step of coordinatization is an explanation how lines $\ell_{a,b}$, $(a, b) \in X_\infty \times X$, may be introduced in the general case.

Coordinatization at a point. Consider a projective plane that contains a line ℓ. Let $(\mathcal{P}, \mathcal{L})$ be the affine plane that is obtained by removing the points of ℓ. Choose two distincts pencils $\mathcal{L}_1$ and $\mathcal{L}_2$ of $(\mathcal{P}, \mathcal{L})$. Each element of $\mathcal{P}$ may be uniquely expressed as $m \cap n$, where $m \in \mathcal{L}_1$ and $n \in \mathcal{L}_2$. By choosing a set X and bijections $\mu_i \colon X \to \mathcal{L}_i$, $i \in \{1, 2\}$, we may identify $\mathcal{P}$ with $X \times X$. The point at the intersection of m and n gets coordinates $(\mu_1(m), \mu_2(n))$. This is called the *grid coordinatization* of $(\mathcal{P}, \mathcal{L})$, with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$.

In the text below it will be assumed that $0 \in X$. The aim is to find a quasigroup operation $\cdot$ upon $X^* = X \setminus \{0\}$, extended by $x0 = 0 = 0x$, for all $x \in X$, in such a way that each line may be labelled naturally as $\ell_{a,b}$, where $a \in X_\infty = X \cup \{\infty\}$ and $b \in X$.

The pencil $\mathcal{L}_1$ will be identified with lines $\ell_a = \ell_{\infty,a} = \{(a,u);\ u \in X\}$, and $\mathcal{L}_2$ with lines $\ell_{0,a} = \{(u,a);\ u \in X\}$, where a runs through X.

Denote by $\mathcal{C}_0$ the set of lines concurrent to $(0,0)$. Choose a bijection $\mu_3\colon X_\infty \to \mathcal{C}_0$ in such a way that $\mu_3(\infty) = \ell_{\infty,0}$ and $\mu_3(0) = \ell_{0,0}$.

Define a binary operation $\cdot$ upon X^* in such a way that $xa = y$ if and only if $(x,y) \in \mu_3(a)$. This is a quasigroup operation, as follows from the fact that if $x, y \in X^*$, then $\mu_3(a)$ intersects both ℓ_x and $\ell_{0,y}$ in exactly one point.

The triple (μ_1, μ_2, μ_3) is called a *coordinatization* of $(\mathcal{P}, \mathcal{L})$ *at point* $(0,0)$, with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$. The operation $\cdot$ may be considered as the *multiplication* induced by this coordinatization.

Assume $a \in X^*$. The line $\{(x,y) \in \mathcal{P};\ y = xa\}$ will be denoted by $\ell_{a,0}$. For each $b \in X$ there exists only one line $s \parallel \ell_{a,0}$ such that $(0,b) \in s$. Denote s by $\ell_{a,b}$. It is now obvious that *every line of* $(\mathcal{P}, \mathcal{L})$ *is equal to a line* $\ell_{a,b}$*, where* $(a,b) \in X_\infty \times X$ *is determined uniquely.*

Coordinatization by loops. The aim now is to show how a coordinatization (μ_1, μ_2, μ_3) may be derived from knowledge of a point $P = (0,0)$, pencils $\mathcal{L}_1$ and $\mathcal{L}_2$ and a bijection $\nu\colon X \to m$, where m is a line, $P \in m$, $m \notin \mathcal{L}_1 \cup \mathcal{L}_2$, $0 \in X$, and $\nu(0) = P$. Let us also assume that X contains a point $1 \neq 0$.

Define $\mu_i\colon X \to \mathcal{L}_i$, $i \in \{1,2\}$, by setting $\mu_i(x) = \ell$ if ℓ intersects m at $\nu(x)$. This yields a grid coordinatization such that m consists of all (a,a), $a \in X$.

Define $\mu_3\colon X_\infty \to \mathcal{C}_0$ in such a way that $\mu_3(a) = \ell_{a,0}$ is the line that intersects the line ℓ_1 at $(1,a)$ for every $a \in X$. Therefore $1a = a$. Furthermore, since $m = \ell_{1,0}$, $a1 = a$ for all $a \in X$.

We have verified that coordinatization at a point may always be chosen in such a way that $(X^*, \cdot, 1)$ is a loop.

Transitivity of perspectivities and coordinatization. The discussion above started from a projective plane from which line ℓ_∞ was removed, yielding thus an affine plane $(\mathcal{P}, \mathcal{L})$.

Suppose that this projective plane is (U, ℓ_∞)-transitive, $U \in \ell$. Choose $\mathcal{L}_1$ to be the pencil of U. Choose also another pencil $\mathcal{L}_2 \neq \mathcal{L}_1$, and consider a coordinatization (μ_1, μ_2, μ_3) at $(0,0)$, with respect to $\mathcal{L}_1$ and a pencil $\mathcal{L}_2 \neq \mathcal{L}_1$.

For $a \in X_\infty$ denote by (a) the pencil $\{\ell_{a,b};\ b \in X\}$. Regard (a) as a point of ℓ_∞. Thus $\mathcal{L}_1 = (\infty) = U$ and $\mathcal{L}_2 = (0)$.

The projective plane is assumed to be $((\infty), \ell_\infty)$-transitive. For each $c \in X$ there thus exists exactly one affine collineation α_c that sends $(0,0)$ upon $(0,c)$, fixes lines ℓ_a, $a \in X$, and retains each pencil. All these collineations form a group that coincides with the group of $((\infty), \ell_\infty)$-collineations. This group is regular upon ℓ_0 (*regular* means transitive and fixed point free). Hence there exists a (unique) group $(X, +, 0)$ such that $\alpha_c(0,d) = (0, c+d)$, for all $c, d \in X$. Each α_c permutes the lines of $\mathcal{L}_2 = (0)$, which means that $\ell_{0,d}$ is sent upon $\ell_{0,c+d}$. An affine point (x,y), which is the intersection of ℓ_x and $\ell_{0,y}$, is thus sent upon the intersection of ℓ_x and $\ell_{0,c+y}$, and that is $(x, c+y)$. Hence $\alpha_c(x,y) = (x, c+y)$ for all $x, y \in X$.

Let a and b be elements of X. The line $\ell_{a,b}$ is the (unique) line parallel to $\ell_{a,0}$ that conntains $(0,b)$. Since $\alpha_b(0,0) = (0,b)$, there has to be $\alpha(\ell_{a,0}) = \ell_{a,b}$. Since $\ell_{a,0} = \{(x,y) \in X \times X;\ y = xa\}$, $\ell_{a,b} = \{(x,y) \in X \times X;\ y = xa + b\}$. We have proved that $((\infty), \ell_\infty)$-transitivity implies the possibility to express lines by means of a quasigroup operation and a group operation, for any coordinatization at point $(0,0)$. This may be recorded as follows:

Proposition. *Consider a projective plane that is (U,ℓ)-transitive, U a point upon line ℓ. Consider the affine plane obtained by removing the line ℓ, and denote by $\mathcal{L}_1$ the pencil of affine lines induced by U. Choose another pencil $\mathcal{L}_2$ and an affine point P. Let (μ_1,μ_2,μ_3) be a coordinatization by X at $P=(0,0)$, with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then there exists weak cartesian group $(X,+,\cdot,0)$ such that the lines $\ell_{a,b}$, $(a,b)\in X_\infty\times X$, are the lines induced by the cartesian group. In particular, $(\infty)=\mathcal{L}_1$ and $(0)=\mathcal{L}_2$.*

Since the coordinatization may always be done by a loop, the ensuing consequence is immediate.

Corollary. *A projective plane may be described by a cartesian group if and only if it is (c,a)-transitive for at least one pair (c,a), c a point upon a line a.*

Coordinatization of a projective plane. Let A, B and C be three noncollinear points of a projective plane. Denote by ℓ_∞ the line connecting A and B, by $\mathcal{L}_1$ the pencil that includes the line AC and by $\mathcal{L}_2$ the pencil that includes the line BC. A coordinatization at $C=(0,0)$ is determined by three bijective images of X, as given by μ_i, $1\le i\le 3$. We have $A=(\infty)$, $B=(0)$ and $C=(0,0)$. It may be thus said that the coordinatization is determined by the (ordered) triple (A,B,C) and by the bijections μ_i, $1\le i\le 3$.

Consider now four points A, B, C and D, none three of which are collinear. Let CD be the line m that is used when coordinating the plane by a loop, and suppose that $D=(1,1)$. The coordinatization is fully determined by a bijection $\nu\colon X\to m$. Note that $\nu(1)=D$. If $\nu'\colon X'\to m$ is a bijection such that $\nu'(1)=D$, then the coordinatization induced by ν' differs only formally. To be exact, the line $\ell'_{a',b'}$ is equal to the line $\ell_{\gamma(a'),\gamma(b')}$, where $\gamma=\nu^{-1}\nu'$. This may be interpreted by saying that the (ordered) *quadruple (A,B,C,D) induces a loop coordinatization up to isomorphism.*

Instead of specifying the bijection ν it is usual to assume that m consists of points (x,x), $x\in X$, where 0 and 1 are two distinct points of X. Under this assumption each line of the projective plane is equal either to ℓ_∞ or to $\ell_{a,b}\cup\{(a)\}$, where $(a,b)\in X_\infty\times X$. Furthermore $m=\ell_{1,1}$, $\ell_{0,x}$ passeses through (x,x) and $(0,x)$, while $\ell_{\infty,x}=\ell_x$ passes through (x,x) and $(x,0)$.

Ternary rings. Here we are mainly concerned with affine planes for which there exist binary operations $\cdot$ and $+$ such that $\ell_{a,b}$ coincides with $\{(x,xa+b);\ x\in X\}$. In general it is possible to introduce a ternary operation T upon X such that $T(u,v,w)=z$ if the line $\ell_{v,w}$ intersects the line ℓ_u at (u,z). In the case of weak cartesian group, $T(u,v,w)=z$ if and only if $z=uv+w$. To allure to this situation $T(u,v,w)$ is often written as $u\cdot v\circ w$. Given a ternary T it is not difficult to give explicit conditions under which T induces an affine plane (while doing this it is usual to assume that X contains both 0 and 1). If T satisfies these conditions, then (X,T) is called a *ternary ring.*

For our purposes the exact form of axioms describing ternary rings is not important (the subject is straigthforward and somewhat boring). However, a warning should be issued that there are three or more ways how a ternary ring is axiomatized. Differences are technical and have no bearing on structural theorems. The approach taken here is close to that of Marshall Hall jr.