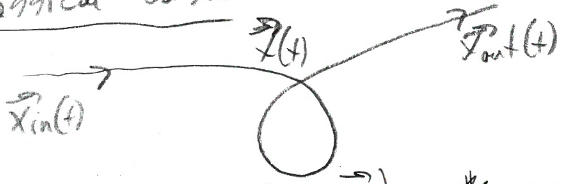
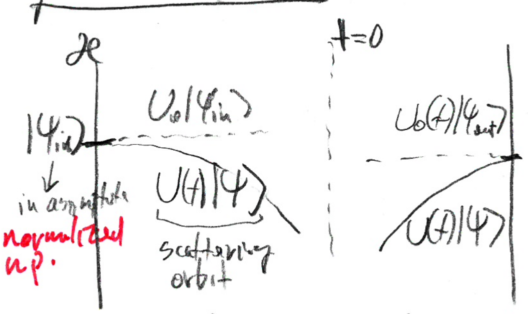
① Formal scattering theory: develops scattering theory by analysis of $\hat{S}$ operator for $\langle\psi_{in}|\hat{S}|\psi_{out}\rangle$

scattering theory comes from elmg. field

classical case



quantum case



- any six numbers $(\vec{a}_{in}, \vec{V}_{in})$ can represent an incoming particle which has an out asymptote too
- not every $\vec{x}(t)$ has in/out asymptotes

$U(t) = e^{-iHt}$ – unitary evolution

- any state from $\mathcal{H}$ can be an in asymptote
- not every $|\psi\rangle$ from $\mathcal{H}$ has asymptotes (bound states)
- only scattering orbits have asymptotes

② What is the amplitude of finding the system in $|\psi_{out}\rangle$ when it was initially prepared in state $|\psi_{in}\rangle$?

- it must be matrix element of some operator:

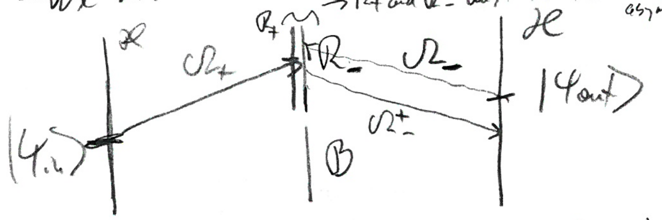
$f_{in\to out} = \langle\psi_{in}|\hat{S}|\psi_{out}\rangle$, $\hat{S}$: scattering operator ... $\langle\psi_{in}|\hat{S}|\psi_{out}\rangle$ ... S-matrix element

- if $|\psi_{in}\rangle$ is any vector in $\mathcal{H}$ (incl. bound) then we must show it corresponds to some scattering orbit, i.e. that it has an out asymptote

⇒ NOW DO ASYMPTOTIC CONDITION: $U(t)|\psi(t)\rangle \xrightarrow{t\to-\infty} U_0(t)|\psi_{in}\rangle$

⇒ Møller operators $\Omega_\pm$: ~~explain they are only~~ isometric (DO THIS LATER SEE [illegible]) map any vector from $\mathcal{H}$ to a scattering orbit (which has in/out asymptotes) ⇒ range of $\Omega_\pm$ excludes bound states

- we have arrived at the following picture: → R_+ and R_- must be the same (asymptotic completeness)



$\langle\chi_-| = \langle\psi_{out}|\Omega_-^+$

amplitude at $t=0$

$\langle\chi_-|\phi_+\rangle = \langle\psi_{out}|\Omega_-^+\Omega_+|\psi_{in}\rangle = \langle\psi_{out}|\hat{S}|\psi_{in}\rangle$

- what is the S-matrix ...
- we need to know that $R_\pm = \mathcal{R}$ (asymptotic completeness)
- we need to prove that $\mathcal{R} \perp \mathcal{B}$ (orthogonality theorem) ⇒ NOW DO OG THEOREM
- alternatively $|\psi_{out}\rangle = \hat{S}|\psi_{in}\rangle$

⇒ we see that $\hat{S}$ maps $\mathcal{H}$ onto $\mathcal{H}$ ⇒ it is unitary $S^+S = 1$

NOW DO UNITARY VS ISOMETRIC OPS.

but $\Omega_\pm$ are not! $\|\Omega_\pm\psi\| = 1 \Leftrightarrow \Omega_\pm^+\Omega_\pm = 1$ but not $\Omega_\pm\Omega_\pm^+ = 1$!

③ the amplitude $\langle\psi_{in}|\hat{S}|\psi_{out}\rangle$ includes the possibility that no interaction took place

(S=1 in absence of all interactions)

⇒ we can split $\hat{S}$ into two parts: $\hat{S} = 1 + \hat{R}$ (scattering contribution)

- the general form of $\hat{S}$ and $\hat{R}$ can be made more precise exploiting conservation of energy

⇒ NOW DO INTERTWINING RELATIONS AND CONSERVATION OF ENERGY

we can write ~~$\hat{R} = \delta(E_p - E_{p'})$~~ $\langle p'|S|p\rangle = \delta(E_{p'}-E_p)\cdot$ remainder

define T-matrix: $\langle\vec{p}'|S|\vec{p}\rangle = \delta(\vec{p}'-\vec{p}) - 2\pi i\,\delta(E_{p'}-E_p)\, t(\vec{p}\to\vec{p}') = \delta(\vec{p}'-\vec{p}) + \frac{i}{2\pi m}\delta(E_{p'}-E_p)\, f(\vec{p}\to\vec{p}')$

from ψ_{out} asymptotes

NOW DO THE OPTICAL THEOREM

Unitary vs isometric operators

unitary operator: maps whole of $\mathcal{H}$ onto whole of $\mathcal{H}$ and preserves the norm.

$$\|U\psi\| = \|\psi\|$$

$$\|U\psi\| = \langle\psi|U^+U|\psi\rangle = \langle\psi|\psi\rangle$$

$$\Rightarrow \boxed{U^+U = 1}$$

~~existence of U^{-1} requires:~~ ~~[illegible]~~ $|\alpha\rangle \neq |\beta\rangle \Rightarrow U|\alpha\rangle \neq U|\beta\rangle$

$\Updownarrow$

– every vector in $\mathcal{H}$ has a unique image

$|\psi\rangle \neq 0 \Rightarrow U|\psi\rangle \neq 0$

since they preserve the norm ($\|U\psi\| = \|\psi\| \neq 0$)

~~[illegible]~~ $\Rightarrow$ unitary operators have inverses.

and $U^{-1} = U^+$ since $U^+U = 1 \quad / U.$

$UU^+U = U \quad / \, |\psi\rangle$ any vector from $\mathcal{H}$

$UU^+ (U|\psi\rangle) = U|\psi\rangle$

– now the crucial point is that U ranges over the whole $\mathcal{H}$ $\Rightarrow U|\psi\rangle = |\psi'\rangle \in \mathcal{H}$

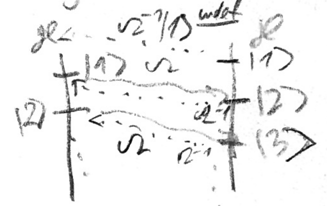
$\Rightarrow \quad UU^+|\psi'\rangle = |\psi'\rangle \quad$ (any $|\psi'\rangle \in \mathcal{H}$)

$$\Rightarrow \boxed{UU^+ = 1}$$

isometric operators (e.g. Møller operators): defined on the whole of $\mathcal{H}$ and preserves the norm

e.g. $\Omega|n\rangle = |n+1\rangle$

$\Downarrow \quad \|\Omega\psi\| = \|\psi\| \Rightarrow \boxed{\Omega^+\Omega = 1}$



but not $\Omega\Omega^+ = 1$!

but $\Omega^{-1}|1\rangle$ is not defined! $\Rightarrow$ I cannot make the last trick $\Omega\Omega^+ \underbrace{\Omega|\psi\rangle} = \underbrace{\Omega|\psi\rangle}$

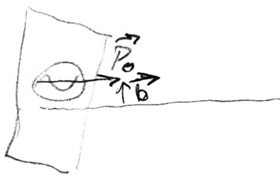
this does not range over whole of $\mathcal{H}$!

Hence $\Omega\Omega^+ = 1$ is only valid in the subspace $R(\Omega)$

$$\Omega^+ = \begin{cases} \Omega^{-1} & \text{on } R(\Omega) \\ 0 & \text{on } R(\Omega)^\perp \end{cases}$$

Calculation of the quantum cross section

- we have as before a particle displaced by an impact par. $\vec{b}$ incident on a target
- now we've formulated the scattering in momentum basis



- we ask what is the probability of observing a particle with momentum in some element of solid angle $d\Omega$ (this is of course equivalent to the spatial picture)!

the cross section is as before: $\sigma(d\Omega \leftarrow \phi) = \int d^2b \; w(d\Omega_{\vec{p}} \leftarrow \phi_{\vec{b}})$ — $\int d^2b$: "sum over all impact pars." — plane perp. to the incident momentum

the probability

$$w(d\Omega \leftarrow \phi) = d\Omega \int_0^\infty dp\, p^2 |\psi_{out}(\vec{p})|^2$$

→ integral over magnitude of p: since we assume a well-defined incident momentum in the incident wp. and energy is conserved we don't get more information by resolving the final state (in energy)

$$\psi_{out}(\vec{p}) = \int d^3p' \langle \vec{p}|S|\vec{p}'\rangle \psi_{in}(\vec{p}'),$$

$$\psi_{in}(\vec{p}) = e^{-i\vec{b}\cdot\vec{p}}\phi(\vec{p}) \qquad \left(e^{-i\vec{b}\cdot\vec{P}} \int d^3p'\, \phi(\vec{p}')\,|\vec{p}'\rangle\right)$$

→ displacement of the wavepacket ... Taylor series for $\psi(\vec{x}-\vec{b}) \to \exp(-\vec{b}\cdot\vec{\nabla})\psi(\vec{x})$

- in terms of the scattering ampl.:

$$\psi_{out}(\vec{p}) = \psi_{in}(\vec{p}) + \frac{i}{2\pi m}\int d^3p'\, \delta(E_p - E_{p'})\, f(\vec{p}' \to \vec{p})\, \psi_{in}(\vec{p}')$$

now do the proof of δ-function integral

- we don't observe in the forward direction $(\vec{p} = \vec{p}_0)$ where $\psi_{in}(\vec{p})$ has a peak ⇒ only the scattered part remains

$$\psi_{out}(\vec{p}) = \frac{i}{2\pi m}\int d^3p'\, \delta(E_p - E_{p'})\, f(\vec{p}' \to \vec{p})\, e^{-i\vec{b}\cdot\vec{p}'}\phi(\vec{p}')$$

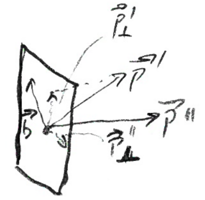
- substitute into the expression for the cross section

$$\sigma(d\Omega \leftarrow \phi) = \frac{d\Omega}{(2\pi m)^2}\int d^2b \int_0^\infty dp\, p^2 \int d^3p'\, \delta(E_p - E_{p'})\, f(\vec{p}' \to \vec{p})\, e^{-i\vec{b}\cdot\vec{p}'}\phi(\vec{p}') \int d^3p''\, \delta(E_p - E_{p''})\, f^*(\vec{p}'' \to \vec{p})\, e^{i\vec{b}\cdot\vec{p}''}\phi^*(\vec{p}'')$$

- first we can do the integral over b:

$$\int d^2b\, e^{+i\vec{b}(\vec{p}'' - \vec{p}')} = \delta_2(\vec{p}_\perp'' - \vec{p}_\perp')\,(2\pi)^2$$

($\vec{p}'' - \vec{p}'$: projection into $\vec{b}$ plane; $\vec{p}_\perp'' - \vec{p}_\perp'$: components in the plane where $\vec{b}$ lives)



- then the 2nd δ function is $\delta(E_p - E_{p''}) = \delta(E_{p'} - E_{p''})$ thanks to the first one

⇒ since $\vec{p}_\perp' = \vec{p}_\perp''$ the δ requirement $\delta(E_{p'} - E_{p''})$ implies $p_\parallel' = \pm p_\parallel''$ but $\phi(\vec{p})$ is peaked about a single momentum $\vec{p}_0$ so

$$p_\parallel' = p_\parallel'' \;\Rightarrow\; \delta_2(\vec{p}_\perp' - \vec{p}_\perp'')\,\delta(E_{p'} - E_{p''}) = \delta_2(\vec{p}_\perp' - \vec{p}_\perp'')\,\delta(E_{p_\parallel'} - E_{p_\parallel''}) = \frac{m}{p_\parallel'}\delta_3(\vec{p}' - \vec{p}'')$$

$$\delta(E_{p_\parallel'} - E_{p_\parallel''}) = \frac{m}{p_\parallel'}\delta(p_\parallel' - p_\parallel'')$$

requirement on magnitudes of the parallel components since the perpendicular momenta have been fixed already

– insert this into the expression – do integral over $\int d b^2$ and $\int d^3_p{}''$

$$\sigma(d\Omega \leftarrow \phi) = \frac{d\Omega}{(2\pi m)^2} (2\pi)^2 \cdot m \int_0^\infty dp\, p^2 \int d^3p' \frac{1}{p'_{\parallel}} \delta(E_p - E_{p'}) \left| f(\vec{p}\,' \to \vec{p}) \cdot \phi(\vec{p}\,') \right|^2$$

– now do the integral over p (using again $\int_0^\infty dp\, g(p)\, \delta(E_p - E_{p'}) = \frac{m}{p'} g(p')$) : $\int_0^\infty dp\, p^2\, \delta(E_p - E_{p'}) = \frac{m}{p'} \cdot (p')^2 = m\, p'$

$$\sigma(d\Omega \leftarrow \phi) = \frac{d\Omega}{m} \cdot m \int d^3p' \frac{p'}{p'_{\parallel}} \left| f(\vec{p}\,' \to \vec{p}) \cdot \phi(\vec{p}\,') \right|^2 \quad \text{with } |\vec{p}| = |\vec{p}\,'|$$

– now we assume that $\phi(\vec{p})$ is very narrow so $\frac{p'}{p'_{\parallel}}$ and f can be replaced by their values at $\vec{p} = \vec{p}_0$

($p_{\parallel}$ is the momentum component parallel to $\vec{p}_0$)

$$\Rightarrow \sigma(d\Omega \leftarrow \phi) = d\Omega \left| f(\vec{p}_0 \to \vec{p}) \right|^2 \cdot \underbrace{\int d^3p' \left| \phi(\vec{p}\,') \right|^2}_{\text{normalization of the incident wp.}} = 1$$

→ σ does not dep. on precise shape of $\phi(p)$

$$\Rightarrow \sigma(d\Omega \leftarrow \vec{p}_0) = \underbrace{\frac{d\sigma}{d\Omega}(\vec{p}_0 \to \vec{p})}_{\text{some factor}} \cdot d\Omega$$

($\vec{p}_0$: the only property of the initial state that is important)

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \left| f(\vec{p}_0 \to \vec{p}) \right|^2}$$

optical theorem : $\mathrm{Im}[f(\vec{p}\to\vec{p})] = \frac{p}{4\pi}\sigma(p)$

- consequence of unitarity of S-matrix ⇒ conservation of probability
  - it is found by looking into the forward direction
  - it can be derived in different ways (TD, TI)
  - $\vec{j}(\vec{r}) \overset{r\to\infty}{\simeq} \vec{j}_{in} + \vec{j}_{out} + \vec{j}_{int}$, $\oint \vec{j}\cdot d\vec{s} = 0$ (conservation of flux)

$$S^\dagger S = 1 \quad (1)$$

$$S = 1 + R \quad (2)$$

R → remainder, i.e. describes transitions due to projectile-target interaction ($S=1$ in absence of all interactions)
– we first make preparatory work and then substitute expressions for scattering amplitude.

Insert (2) into (1) and find

$$(1+R^\dagger)(1+R) = 1$$

$$R + R^\dagger + R^\dagger R = 0 \Rightarrow R + R^\dagger = -R^\dagger R$$

Insert a complete set of states ($1 = \int d^3p''\,|p''\rangle\langle p''|$) and take matrix elements $\langle p'|\ldots|p\rangle$

$$\langle p'|R|p\rangle + \underbrace{\langle p'|R^\dagger|p\rangle}_{=\langle p|R|p'\rangle^*} = -\langle p'|R^\dagger R|p\rangle$$

$$\langle p'|R|p\rangle + \langle p|R|p'\rangle^* = -\int d^3p''\,\underbrace{\langle p'|R^\dagger|p''\rangle}_{=\langle p''|R|p'\rangle^*}\langle p''|R|p\rangle$$

$$\Rightarrow \langle p'|R|p\rangle + \langle p|R|p'\rangle^* = -\int d^3p''\,\langle p''|R|p'\rangle^*\langle p''|R|p\rangle$$

We have defined T-matrix elements as

$$\langle p'|S|p\rangle = \delta(\vec{p}'-\vec{p}) - 2\pi i\,\delta(E_{p'}-E_p)\,t(\vec{p}\to\vec{p}') = \delta(\vec{p}'-\vec{p}) + \underbrace{\frac{i}{2\pi m}\delta(E_{p'}-E_p)\,f(\vec{p}\to\vec{p}')}_{\langle p'|R|p\rangle}$$

$$f(\vec{p}\to\vec{p}') = -(2\pi)^2 m\, t(\vec{p}\to\vec{p}')$$

$$\Rightarrow \langle p'|R|p\rangle = \frac{i}{2\pi m}\delta(E_{p'}-E_p)\,f(\vec{p}\to\vec{p}')$$

$$\frac{i}{2\pi m}\delta(E_{p'}-E_p)\left(f(\vec{p}\to\vec{p}') - f^*(\vec{p}\to\vec{p}')\right) = +\left(\frac{i}{2\pi m}\right)^2\cdot\int d^3p''\,\delta(E_{p''}-E_{p'})\,f^*(\vec{p}'\to\vec{p}'')\cdot\delta(E_{p''}-E_p)\,f(\vec{p}\to\vec{p}'')$$

(+ : c.c. of $\langle p''|R|p'\rangle^* \rightsquigarrow \left(\frac{i}{2\pi m}\right)^* = -\frac{i}{2\pi m}$; $\delta(E_{p''}-E_p)$ ⇓ $E_{p''}=E_p$, $\delta(E_{p''}-E_{p'}) = \delta(E_p-E_{p'})$)

$$\frac{i}{2\pi m}\delta(E_{p'}-E_p)\left[f(\vec{p}\to\vec{p}') - f^*(\vec{p}\to\vec{p}')\right] = \left(\frac{i}{2\pi m}\right)^2\delta(E_{p'}-E_p)\cdot\int d^3p''\,\delta(E_{p''}-E_p)\cdot f^*(\vec{p}'\to\vec{p}'')\,f(\vec{p}\to\vec{p}'')$$

$$f(\vec{p}\to\vec{p}') - f^*(\vec{p}\to\vec{p}') = \frac{i}{2\pi m}\int d^3p''\,\delta(E_{p''}-E_p)\,f^*(\vec{p}'\to\vec{p}'')\,f(\vec{p}\to\vec{p}'')$$

Now set $\boxed{\vec{p}'=\vec{p}}$ (we look at scattering in the forward direction)

$$2\,\mathrm{Im}[f(\vec{p}\to\vec{p})] = \frac{1}{2\pi m}\underbrace{\int_0^\infty dp''\,(p'')^2\,\delta(E_{p''}-E_p)}\int d\Omega_{p''}\,|f(\vec{p}\to\vec{p}'')|^2$$

$$\int_0^\infty dp''\,(p'')^2\,\delta\!\left(\tfrac{1}{2m}[(p'')^2-p^2]\right) = \frac{m}{p}\,\delta(p''-p)\,(p'')^2$$

$$2\,\mathrm{Im}[f(\vec{p}\to\vec{p})] = \frac{1}{2\pi m}\underbrace{\int_0^\infty dp''\,(p'')^2\,\frac{m}{p}\,\delta(p''-p)}_{m\cdot p}\cdot\int d\Omega_{p''}\,|f(\vec{p}\to\vec{p}'')|^2$$

$$2\,\mathrm{Im}[f(\vec{p}\to\vec{p})] = \frac{1}{2\pi m}\cdot mp\underbrace{\int d\Omega_{p''}\,|f|^2}_{\sigma(p)}$$

$$\boxed{\mathrm{Im}[f(\vec{p}\to\vec{p})] = \frac{p}{4\pi}\int d\Omega_p\,|f|^2}$$

scattering takes away amplitude in the forward direction into all directions and hence forward amplitude must be proportional to the total cross section.

Last week:

$\langle \psi_{in} | \hat{S} | \psi_{out} \rangle \rightsquigarrow \langle \vec{p}\,' | \hat{S} | \vec{p} \rangle = \delta(\vec{p}\,'-\vec{p}) - 2\pi i\, \delta(E_{p'}-E_p)\, t(\vec{p}\to\vec{p}\,') = \delta(\vec{p}\,'-\vec{p}) + \frac{i}{2\pi m}\, \delta(E_{p'}-E_p)\, f(\vec{p}\to\vec{p}\,')$

(proper states; momentum basis (improper states); general structure of S-matrix; t describes scattering (effect of the interaction))

we showed that: $\frac{d\sigma}{d\Omega} = |f(\vec{p}_0\to\vec{p})|^2 \Rightarrow f$ is a scattering amplitude

$$t(\vec{p}_0\to\vec{p}) = -(2\pi)^2 m\, f(\vec{p}_0\to\vec{p})$$

How do we calculate t and/or f? ... We know from TI theory (recall L-S equation) that: $t(\vec{p}_0\to\vec{p}) = \langle \vec{p}\,' | T | \vec{p}_0 \rangle = -(2\pi)^2 m \langle \vec{p}\,' | V | \psi^{(+)}_{\vec{p}_0} \rangle$,

where $T = V + VGV$ or $T = V + VG_0T$

A) We will establish the connection $t(\vec{p}_0\to\vec{p}) = \langle \vec{p}\,'|T|\vec{p}_0\rangle$ by a direct calculation of $\langle \chi | S | \phi \rangle$ from the TD theory

B) We will explicitly show what are the b.c. on the $|\psi^{(\pm)}_{\vec{p}_0}\rangle$ ($|p\pm\rangle$) and what is their role in representing the proper scattering states (we have showed earlier that $\psi(t) = \int d^3k\, \phi(\vec{k})\, \psi^{(+)}_{\vec{k}}(\vec{r})\, e^{-iE_k t}$ represents time evolution of a scattering event)

→ We will need a few basic results concerning Green's functions:

- we met Green's functions first when studying L-S equation:

$$[\Delta_r + k^2]\, G_0(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')$$
$$[\Delta_r + k^2 - U]\, G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')$$

- these are coordinate representations of the operator equations:

$$(z-H_0)\,G_0(z) = 1 \quad \dots (z = k^2) \text{ and } z\in\mathbb{C}$$
$$(z-H)\,G(z) = 1$$

insert a complete set of states $|E, l, m\rangle$: $\mathbb{1} = \int\sum |E,l,m\rangle\langle E,l,m|$; $H_0|E,l,m\rangle = E|E,l,m\rangle$, $L^2 \dots = l(l+1)\dots\rangle$, $L_z \dots = m \dots\rangle$

Formally, I can express $G_0(z) = (z-H)^{-1}\cdot\mathbb{1} = \int_0^\infty dE \sum_{lm} \frac{|E,l,m\rangle\langle E,l,m|}{z-E}$ ⇒ analytic function of z except for $z\in\mathbb{R}$ — branch cut on $z\in[0,\infty)$

- similarly for $G(z)$ ... analytic except of bound states and continuum

We also see that $\boxed{G(z)^\dagger = G(z^*)}$ since $H = H^\dagger$

Reminder: L-S equation for $G(z)$: $G(z) = G_0 + G_0 V G(z)$

define T-operator: $T(z) = V + VG(z)\cdot V$

(i.e. this is a different route than before)

and $\boxed{T(z)^\dagger = T(z^*)}$

multiply $T(z)$ by $G^0(z)$ from left/right

$$\Rightarrow \boxed{G_0(z)\,T(z)} = G_0(z)V + G_0(z)\,VG(z)\,V = [G_0(z) + G_0(z)\,VG(z)]\cdot V = \boxed{G(z)\cdot V}$$

Similarly: $T(z)\cdot G_0(z) = VG(z)$

Now we will prove that:

$$\langle p'|S|p\rangle = \delta(\vec{p}'-\vec{p}) - 2\pi i\,\delta(E_{p'}-E_p)\cdot\langle p'|T|p\rangle, \text{ where } T = V + VGV$$

– we start with $\langle\chi|S|\phi\rangle$ for proper states $|\chi\rangle$ and $|\phi\rangle$

order of limits doesn't matter... I can do them together

$$\langle\chi|S|\phi\rangle = \langle\chi|\Omega_-^{+}\Omega_+|\phi\rangle = \lim_{\substack{t\to+\infty\\ t'\to-\infty}} \langle\chi|(e^{iH_0t}e^{-iHt})(e^{iHt'}e^{-iH_0t'})|\phi\rangle = |t'=-t| = \lim_{t\to\infty}\langle\chi|e^{iH_0t}e^{-iHt}e^{iHt}e^{iH_0 t}|\phi\rangle =$$

$$= \lim_{t\to\infty}\langle\chi|\underbrace{e^{iH_0t}e^{-2iHt}e^{iH_0t}}|\phi\rangle = \langle\chi|1 - i\int_0^\infty [e^{iH_0t}Ve^{-2iHt}e^{iH_0t} + e^{iH_0t}e^{-2iHt}Ve^{iH_0t}]|\phi\rangle =$$

$\hookrightarrow \hat{S}$ at $t=0$ $= 1$

$$= \int\frac{d}{dt}\,dt \rightsquigarrow \frac{d}{dt}\left[e^{iH_0t}e^{-2iHt}e^{iH_0t}\right] = iH_0 t\left[e^{iH_0t}e^{-2iHt}e^{iH_0t}\right] + e^{iH_0t}\left[-2iHe^{-2iHt}e^{iH_0t} + e^{-2iHt}iH_0e^{iH_0t}\right] =$$

$= -i-i$

$$= -i\left[e^{iH_0t}Ve^{-2iHt}e^{iH_0t} + e^{iH_0t}e^{-2iHt}Ve^{iH_0t}\right]$$

– I want to calculate $\langle p'|S|p\rangle$ but for the improper states the integral is not convergent (asymptotic condition require spreading wavepackets which the plane waves are not) ⇒ introduce a damping factor first $(e^{-\varepsilon t})$:

$$\langle\chi|\hat{S}|\phi\rangle = \langle\chi|\phi\rangle - i\lim_{\varepsilon\to0}\int_0^\infty dt\, e^{-\varepsilon t}\langle\chi|[\dots]|\phi\rangle$$

$\varepsilon > 0$

... this integral converges absolutely and uniformly and so is unchanged by the limiting process (adiabatic switching theorem)

– now I can replace $|\chi\rangle \to |\vec{p}'\rangle$ and $|\phi\rangle\to|\vec{p}\rangle$

$$\langle\vec{p}'|\hat{S}|\vec{p}\rangle = \delta(\vec{p}'-\vec{p}) - i\lim_{\varepsilon\to0+}\int_0^\infty dt\, e^{-\varepsilon t}\langle\vec{p}'|e^{iH_0t}Ve^{-2iHt}e^{iH_0t} + e^{iH_0t}e^{-2iHt}Ve^{iH_0t}|\vec{p}\rangle = \left|\begin{matrix}H|\vec{p}\rangle = E_p|\vec{p}\rangle\\ H|\vec{p}'\rangle = E_{p'}|\vec{p}'\rangle\end{matrix}\right| =$$

$$= \delta(\vec{p}'-\vec{p}) - i\lim_{\varepsilon\to0+}\int_0^\infty dt\, e^{-\varepsilon t}\langle\vec{p}'|Ve^{i(E_{p'}-2H+E_p)t} + e^{i(E_{p'}-2H+E_p)t}V|\vec{p}\rangle = |-\varepsilon = i\cdot i\varepsilon| =$$

$$= \delta(\vec{p}'-\vec{p}) - i\lim_{\varepsilon\to0+}\int_0^\infty dt\,\langle\vec{p}'|Ve^{i(E_{p'}+E_p-2H+i\varepsilon)t} + e^{i(E_{p'}+E_p-2H+i\varepsilon)t}V|\vec{p}\rangle = \left|\int_0^\infty dt\, e^{i(E_{p'}+E_p-2H+i\varepsilon)t} = \frac{1}{i}(E_{p'}+E_p-2H+i\varepsilon)^{-1}\cdot\left[e^{i(E_{p'}+E_p-2H+i\varepsilon)t}\right]_0^\infty\right.$$

redefine ε

$$= \text{there is no contribution at } t\to\infty \text{ due to } e^{-\varepsilon t}: -\frac{1}{i}(E_{p'}+E_p-2H+i\varepsilon)^{-1} = +\frac{i}{2}\left(\frac{E_{p'}+E_p}{2} - H + 2i\varepsilon\right)^{-1} = \frac{i}{2}G\left(\frac{E_{p'}+E_p}{2}+i\varepsilon\right) \Bigg| =$$

$$= \delta(\vec{p}'-\vec{p}) - i\lim_{\varepsilon\to0+}\left[\langle\vec{p}'|V\frac{i}{2}G\left(\frac{E_{p'}+E_p}{2}+i\varepsilon\right) + \frac{i}{2}G\left(\frac{E_{p'}+E_p}{2}+i\varepsilon\right)V|\vec{p}\rangle\right] = \delta(\vec{p}'-\vec{p}) + \frac{1}{2}\lim_{\varepsilon\to0+}\langle\vec{p}'|VG\left(\frac{E_{p'}+E_p}{2}+i\varepsilon\right) + G(\dots)V|\vec{p}\rangle$$

- now we make use of the relations $VG(z) = G_0(z)T(z)$ and $G(z)V = T(z)G_0(z)$

$$\Rightarrow \langle p'|S|p\rangle = \delta(\vec{p}\,'-\vec{p}) + \frac{1}{2}\lim_{\varepsilon\to0+} \langle \vec{p}\,'| G_0(z)T(z) + T(z)G_0(z)|\vec{p}\rangle = \Big| \; G_0(z) = (z-H_0)^{-1} \Rightarrow \langle \vec{p}|G_0(z) = \langle\vec{p}|\frac{1}{z-E_p'} \Big| =$$

(put it in the denominator)

$$= \delta(\vec{p}\,'-\vec{p}) + \lim_{\varepsilon\to0+} \langle\vec{p}\,'|\frac{1}{E_p'+E_p-2E_p'+i\varepsilon}T(z) + T(z)\frac{1}{E_p'+E_p-2E_p+i\varepsilon}|\vec{p}\rangle = \delta(\vec{p}\,'-\vec{p}) + \lim_{\varepsilon\to0+}\left(\frac{1}{E_p-E_p'+i\varepsilon}+\frac{1}{E_p'-E_p+i\varepsilon}\right)\langle\vec{p}\,'|T\left(\frac{E_p+E_p'}{2}+i\varepsilon\right)|\vec{p}\rangle$$

$$z = \frac{E_p'+E_p}{2}+i\varepsilon \qquad \lim_{\varepsilon\to0+}\left(\dots\right) = -2\pi i\,\delta(E_p-E_p')$$

$$= \delta(\vec{p}\,'-\vec{p}) - 2\pi i\,\delta(E_p-E_p')\cdot\langle\vec{p}\,'|T\left(\frac{E_p+E_p}{2}+i\varepsilon\right)|\vec{p}\rangle = \delta(\vec{p}\,'-\vec{p}) - 2\pi i\,\delta(E_p-E_p')\langle\vec{p}\,'|T(E_p+i\varepsilon)|\vec{p}\rangle$$

$$\langle\vec{p}\,'|T(E_p+i\varepsilon)|\vec{p}\rangle = f(\vec{p}\to\vec{p}\,') \qquad T = V+VGV$$

→ we have proved that $f(\vec{p}\to\vec{p}\,')$ is the matrix element of T-operator

- now we will show how to calculate it using the stationary scattering states, i.e. $\langle\vec{p}\,'|T|\vec{p}\rangle = \langle\vec{p}\,'|V|\psi^{(+)}_{\vec{p}}\rangle$

(insert $1 = \int d^3p\,|p\rangle\langle p|$)

$$|\phi\pm\rangle = \Omega_\pm|\phi\rangle = |\phi\rangle + i\lim_{\varepsilon\to0}\int_0^{\mp\infty}dt\, e^{\pm\varepsilon t}U(t)^+VU_0(t)|\phi\rangle = |\phi\rangle + i\lim_{\varepsilon\to0+}\int d^3p\int_0^{\mp\infty}dt\,e^{\pm\varepsilon t}e^{iHt}Ve^{-iH_0t}|\vec{p}\rangle\langle\vec{p}|\phi\rangle =$$

$$\langle\vec{p}|\phi\rangle = \phi(\vec{p})$$

$$= |\phi\rangle + i\lim_{\varepsilon\to0+}\int d^3p\int_0^{\mp\infty}dt\,e^{-i(\pm i\varepsilon+E_p-H)t}V\,\phi(p)|p\rangle = |\phi\rangle + \lim_{\varepsilon\to0+}\int d^3p\,G(E_p\pm i\varepsilon)V|p\rangle\phi(p)$$

expand ϕ in momentum basis: $|\phi\rangle = \int d^3p\,\phi(p)|\vec{p}\rangle$

$$\Rightarrow \text{LHS:} \quad \Omega_\pm|\phi\rangle = \int d^3p\,\phi(\vec{p})\Omega_\pm|\vec{p}\rangle = \int d^3p\,\phi(\vec{p})|\vec{p}\pm\rangle$$

$$\text{RHS:} \quad |\phi\rangle + \lim_{\varepsilon\to0+}\int d^3p\,G(E_p\pm i\varepsilon)V\phi(\vec{p}) = \int d^3p\,\phi(\vec{p})\left(|\vec{p}\rangle + \lim_{\varepsilon\to0+}G(E_p\pm i\varepsilon)V|\vec{p}\rangle\right)$$

$$\Rightarrow \int d^3p\,\phi(p)|p\pm\rangle = \int d^3p\,\phi(p)\left[|\vec{p}\rangle + \lim_{\varepsilon\to0+}G(E_p\pm i\varepsilon)V|\vec{p}\rangle\right)$$

this must be valid for any $\phi(\vec{p})$ (normalized)

$$\boxed{|p\pm\rangle = |\vec{p}\rangle + G(E_p\pm i\varepsilon)V|\vec{p}\rangle = (1+G(E_p\pm i\varepsilon)V)|\vec{p}\rangle}$$

⇒ interpretation of $|p\pm\rangle$ as representing incoming and scattered wave is only accurate when smeared over $\phi(\vec{p})$:

$$U(t)|\psi\rangle = U(t)\Omega_+|\psi_{in}\rangle = \int d^3p\,\phi(\vec{p})\,U(t)\Omega_+|\vec{p}\rangle$$

$$\psi = \Omega_+|\psi_{in}\rangle \qquad \Omega_+|\vec{p}\rangle = |p+\rangle$$

We can now use $|p\pm\rangle$ to calculate $\langle \vec{p}' | T(E_p + i\varepsilon) | \vec{p}\rangle$ since

$$T(E_p+i\varepsilon)|\vec{p}\rangle \overset{T=V+VGV}{=} V + VG(E_p+i\varepsilon)V|\vec{p}\rangle = V\underbrace{\left(1+G(E_p+i\varepsilon)V\right)|\vec{p}\rangle}_{|p+\rangle}$$

$$-\frac{1}{4\pi}\cdot\frac{e^{ip|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

$$\Rightarrow \quad f(\vec{p}\to\vec{p}') = \langle \vec{p}' | T | \vec{p}\rangle = \langle \vec{p}' | V | \vec{p}+\rangle$$

we know that $|p+\rangle$ satisfies outgoing b.c. since $|p+\rangle = (1+G(E_p+i\varepsilon)V)|\vec{p}\rangle$

furthermore since $GV = G_0T \Rightarrow$ $|p+\rangle = (1+G_0T)|\vec{p}\rangle = |\vec{p}\rangle + G_0V|\vec{p}+\rangle$ L-S equation for $|p+\rangle$

$\Rightarrow |p+\rangle$ satisfies Schr. eq. $H|p+\rangle = E_p|p+\rangle$

we can see it also formally:

intertwining rel.: $H\Omega_\pm = \Omega_\pm H_0$

$$H|p\pm\rangle = H\Omega_\pm|p\rangle = \Omega_\pm H_0|p\rangle = \Omega_\pm E_p|p\rangle = E_p|p\pm\rangle$$

$\Rightarrow$ the scattering amplitude $\langle \vec{p}'|T|\vec{p}\rangle = \langle \vec{p}'|V|\vec{p}+\rangle$ is obtained solving $H|\vec{p}+\rangle = E_p|\vec{p}+\rangle$ with b.c. given by $G^{(+)}$

equivalently

$$\langle p'|T(E_p+i\varepsilon) = \langle T^\dagger \vec{p}'| \overset{T^\dagger = T(z^*)}{=} \langle T(E_p-i\varepsilon)\vec{p}'| \overset{T=V+VGV}{=} \langle \vec{p}'-|V$$

$$U(t)|\psi\rangle = \int d^3p\, \phi(p)\, \underbrace{U(t)|p+\rangle}_{e^{-iE_pt}|p+\rangle}$$

$$\Rightarrow \quad \langle \vec{p}'|T(E_p+i\varepsilon)|\vec{p}\rangle = \underline{\langle \vec{p}'-|V|p\rangle}$$

$\Rightarrow |p\pm\rangle$ contain equivalent information but $|p+\rangle$ when smeared over $\phi(\vec{p})$ has the physical interpretation of a particle with a well-defined momentum at $t \gg 0$ and is a complicated function at $t \ll 0$, while $|p-\rangle$ has well-defined momentum incoming with a well-defined momentum at $t \ll 0$.

for sph.-symmetric potentials: $[S,H_0]=0,\ [S,L^2]=0,\ [S,L_3]=0$ (S-matrix is diagonal)

$$\langle E', l', m'|S|E, l, m\rangle = \delta(E'-E)\delta_{l,l'}\delta_{m,m'}\underbrace{S_l(E)}_{=e^{2i\delta_l(E)}}$$

we know that $\langle \vec{p}'|V|\vec{p}+\rangle = \frac{1}{2ip}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l}-1\right)P_l(\underbrace{\cos\theta}_{\text{angle between } \vec{p}', \vec{p}})$, δ_l obtained by solving $\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + p^2\right]u_l(r) = 0$ with b.c. $u_l(r)=0,\ r=0$.