NMAI057 – Linear algebra 1

Tutorial 12 – with solutions

Linear maps – image, kernel, and isomorphism

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Problem 1. Decide and justify whether the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f(x, y, z) = (x + y - 2z, y - z, x - y)^T$$

is in isomorphism of $\mathbb{R}^3$ onto itself (so-called automorphism).

Solution:
Isomorphism is a linear map which is also bijective. We compute the dimension of the kernel of the map (it must be zero for the map to be injective) and the dimension of the image of the map (it must be equal to the dimension of the range for the map to be surjective).

We use the fact that the dimension of f is equal to the dimension of any matrix of f w.r.t. arbitrary basis and, similarly, the dimension of the image of f is equal the dimension of the column space of any matrix of f w.r.t. arbitrary basis. First, we construct the matrix of f w.r.t. the standard basis K of $\mathbb{R}^3$ (as noted above, any basis would do):

$$[f]_{K,K} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Using Gaussian elimination, we find REF of the above matrix :

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the dimension of the kernel of the matrix is one and, thus, the map is not injective. Indeed, $f(0, 0, 0) = (0, 0, 0)^T = f(1, 1, 1)$.

The dimension fo the column space equals the rank of the matrix which is two and, thus, the dimension of the image is also two. Thus, the function is not surjective. Similarly, we could verify that the vector $(0, 0, 1)^T$ is not an element of the image of f (via identical Gaussian elimination with the considered vector as the right hand side of the corresponding linear system).

Problem 2. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by the images of basis B:

$$f(2, 1, 1) = (1, 2, 3)^T,$$
$$f(1, 3, 5) = (3, 2, 1)^T,$$
$$f(7, 1, 4) = (1, 1, 1)^T.$$

Decide and justify whether:

(a) f is injective – if not then find distinct vectors $u, v \in \mathbb{R}^3$ such that $f(u) = f(v)$,

(b) f is surjective (onto) – if not then find a vector without a preimage, i.e., $u \in \mathbb{R}^3$ such that for all $v \in \mathbb{R}^3$ it holds that $f(v) \neq u$.

Compute the dimension and find a basis for both the image and kernel of f.

<u>Solution</u>:

Injectivity and the dimension of kernel: First, we decide whether the map is injective. If it is not injective then there exist distinct $u, v \in \mathbb{R}^3$ (from the domain) such that $f(u) = f(v)$. Let's express this observation using a matrix of the map:

$$\begin{aligned} [f]_{B,A}[u]_B &= [f]_{B,A}[v]_B, \\ [f]_{B,A}[u]_B - [f]_{B,A}[v]_B &= \vec{0}, \\ [f]_{B,A}([u]_B - [v]_B) &= \vec{0}, \end{aligned}$$

where $[f]_{B,A}$ is the matrix of f w.r.t. the bases A, B and $[u]_B, [v]_B$ are the coordinates of vectors u, v w.r.t. B. Thus, $[f(u)]_A = [f]_{B,A}[u]_B$. Therefore, if the map is injective then its kernel contains only the vector $\vec{0}$.

Given the images of the basis vectors from B, we can easily construct the matrix of f w.r.t. the basis B and the standard basis K:

$$[f]_{B,K} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}.$$

Using the Gaussian elimination, we compute the dimension of its kernel:

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 \\ 0 & -4 & -1 \\ 0 & -8 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that the dimension of the kernel of the matrix is one as all the solutions for the corresponding homogeneous system are: $\left\{(-\frac{1}{4}t, -\frac{1}{4}t, t)^T \mid t \in \mathbb{R}\right\}$. The map is not injective. Indeed, we can chose the vector with coordinates $[u]_B = (1, 1, -4)^T$, i.e., $u = (31, 8, 8)^T$, that will be mapped to the all-zero vector (similarly to the all-zero vector)

$$f(0, 0, 0) = (0, 0, 0)^T = f(31, 8, 8).$$

Note that the vectors in kernel of $[f]_{B,K}$ correspond to the *coordinates* of vectors in the kernel of f w.r.t. the basis B. To express the vector u w.r.t. the standard basis, we had to take the corresponding linear combination of the vectors from B.

Since the dimension of the kernel is one, its basis is for example the vector $u = (31, 8, 8)^T$.

Surjectivity and the dimension of the image: Every vector in the image is a linear combination of the columns of $[f]_{B,K}$. For example, there exists a vector $a \in \mathbb{R}^3$ such that $f(a) = (1, 2, 3)^T$ (w.r.t. the standard basis). Note that

a is exactly the the first basis vector $(2,1,1)^T$ from B, for which $[(2,1,1)^T]_B = (1,0,0)^T$.

The above Gaussian elimination gives also that the dimension of the image of f equals two and that a basis of the column space are the first two columns $(1,2,3)^T, (3,2,1)^T$. Since the dimension of the image is two and the dimension of the range is three, we can conclude that f is not surjective.

To get a vector which is outside of the image of f, we can extend the basis of the image to a basis of the whole space $\mathbb{R}^3$. The added vector will not have a preimage under f. For example, if we try to extend the basis using the vectors of standard basis the we get the vector $(0,0,1)^T$.

Problem 3. Let $f\colon U \to V$ and $g\colon V \to W$ be isomorphisms. Prove that their composition $g \circ f\colon U \to W$ is also an isomorphism. In particular, show that:

(a) $g \circ f$ is injective,

(b) $g \circ f$ is surjective.

Solution:

(a) We can assume that:

$$\forall u_1, u_2 \in U\colon u_1 \neq u_2 \Rightarrow f(u_1) \neq f(u_2),$$
$$\forall v_1, v_2 \in V\colon v_1 \neq v_2 \Rightarrow g(v_1) \neq g(v_2).$$

We need to show:

$$\forall u_1, u_2 \in U\colon u_1 \neq u_2 \Rightarrow g(f(u_1)) \neq g(f(u_2)).$$

For all distinct $u_1, u_2 \in U\colon u_1 \neq u_2$, it holds that $f(u_1) \neq f(u_2)$ (f is injective). Since g is injective and $f(u_1), f(u_2) \in U\colon f(u_1) \neq f(u_2)$, it follows that $g(f(u_1)) \neq g(f(u_2))$.

(b) Only hint: for an arbitrary $w \in W$ we first find its preimage v w.r.t. g and then the preimage of v w.r.t. f.

Problem 4. Decide and justify whether the following vector spaces are isomorphic:

(a) $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^4$,

(b) $\mathbb{R}^4$ and $\mathcal{P}^3$ (the space of all real polynomials of degree at most three),

(c) $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{n\times m}$,

(d) $\mathbb{R}^n$ over $\mathbb{R}$ and $\mathbb{C}^n$ over $\mathbb{R}$,

(e) $\mathbb{R}^2$ and $\{v \in \mathbb{R}^4 \mid x_1 + x_2 = x_3 + x_4 = 0\}$,

(f) the space of all real polynomials and the space of all real sequences,

(g) $\mathbb{R}^4$ and the space of all linear maps (forms) $f\colon \mathbb{R}^4 \to \mathbb{R}$.

Solution:
We verify whethere there exists an isomorphism of the given spaces.

(a) $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^4$

Yes, the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

is an isomorphism of the given spaces. .

(b) $\mathbb{R}^4$ and $\mathcal{P}^3$ (the space of all real polynomials of degree at most three),

Yes, every real polynomial is uniquely represented by the four real coefficients $p_1, \ldots, p_4$ such that $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$. Conversely, every vector from $\mathbb{R}^4$ can be mapped to the corresponding real polynomial of degree at most three.

(c) $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{n\times m}$

Yes, the most natural isomorphism is the transposition operation.

(d) $\mathbb{R}^2$ a $\{v \in \mathbb{R}^4 \mid x_1 + x_2 = x_3 + x_4 = 0\}$

Yes, for example using the map

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ -a \\ b \\ -b \end{pmatrix}.$$

(e) the space of all real polynomials and the space of all real sequences,

No. Note that there are no polynomials of infinite degree but there are infinite real sequences, e.g. $a_n = 1$ for all $n \in \mathbb{N}$.

(f) $\mathbb{R}^4$ and the space of all linear maps (forms) $f\colon \mathbb{R}^4 \to \mathbb{R}$.

Yes, we can map any $u \in \mathbb{R}^4$ to the map $f(v) = u^T v$. Conversely, every linear form $f\colon \mathbb{R}^4 \to \mathbb{R}$ can be represented via a matrix with a single row and four columns (which is isomorphic to a column vector).

Problem 5. For the linear map $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined as $A \mapsto A - A^T$, decide and justify which of the given vectors are elements of the image of f and which are elements of the kernel of f:

(a)

$$I_2,$$

(b)

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

(c)

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

(d)
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

<u>Solution</u>:

(a) I_2
It is an element of the kernel since $I_2 - I_2^T = 0$ (the zero matrix). It is not in the image since all matrices in the image have zeros on the main diagonal.

(b)
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
Both in the kernel and the image (it is mapped to itself).

(c)
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
It is in the kernel but not in the image.

(d)
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
It is not in the kernel but it is the image of (among others) of the matrix:
$$\begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

Problem 6. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Denote $f^1 = f$, $f^2 = f \circ f, \ldots, f^n = f \circ f^{n-1}$. Prove that $\mathrm{Ker}(f^n) \subseteq \mathrm{Ker}(f^{n+1})$.

<u>Solution</u>:
First, we interpret the map f using $A = [f]_{K,K}$ (we use A to simplify the notation below). We have $\forall v \in \mathbb{R}^n\colon f(v) = Av$. Moreover, $\forall v \in \mathbb{R}^n\colon f^n(v) = A^n v$.

If $v \in \mathrm{Ker}(f^n)$ then $f^n(v) = \vec{0}$ and, therefore, $A^n v = \vec{0}$. It follows that
$$A^{n+1}v = A(A^n v) = A\vec{0} = \vec{0}.$$
Thus, $v \in \mathrm{Ker}(f^n) \Rightarrow v \in \mathrm{Ker}(f^{n+1})$ and we can conclude that $\mathrm{Ker}(f^n) \subseteq \mathrm{Ker}(f^{n+1})$.

Problem 7. Decide and justify whether the given linear map is injective and surjective:

(a) $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^3$ defined as $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b+c, a+b, a)^T$,

(b) $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ defined as $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b+c+d, a+b+c, a+b, a)^T$,

(c) $f\colon \mathcal{P}^2 \to \mathbb{R}^4$ defined as $f(ax^2+bx+c) = (a+b, 2b-c, a-b+c, a+b)^T$,
(d) $f\colon \mathcal{P}^2 \to \mathbb{R}^3$ defined as $f(ax^2+bx+c) = (a+b, 2b-c, a-b+c)^T$,
(e) $f\colon \mathcal{P}^2 \to \mathbb{R}^3$ defined as $f(ax^2+bx+c) = (a+b, 2b-c, a-b+2c)^T$.

Solution:

(a) It is surjective but not injective.
(b) It is an isomorphism.
(c) It is neither surjective (the first and last coordinate of any image are equal) nor injective ($f(x^2-x-2) = (0,0,0,0)^T$).

Problem 8. Prove that for all $A \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{p\times n}$,

$$\dim(\operatorname{Ker}(A) \cap \mathcal{C}(B)) = \operatorname{rank}(B) - \operatorname{rank}(AB),$$

where $\mathcal{C}(B)$ denotes the column space of B.

Solution:
Let $\dim(\operatorname{Ker}(A) \cap \mathcal{C}(B)) = k$ and $v_1, v_2, \ldots, v_k$ be its basis. We extend it to the basis of $\mathcal{C}(B)$ using vectors $w_1, w_2, \ldots, w_\ell$, i.e., $\operatorname{rank}(B) = \dim(\mathcal{C}(B)) = k + \ell$. The statement is then equivalent to proving that $\operatorname{rank}(AB) = \dim(\mathcal{C}(AB)) = \ell$.

The main observation here is that we can describe the column space of AB solely by understanding where A maps the basis $v_1, \ldots, v_k, w_1, \ldots, w_\ell$ of $\mathcal{C}(B)$. First, the vectors $v_1, \ldots, v_k$ are mapped to the zero vector. To conclude the proof, we show that $Aw_1, Aw_2, \ldots, Aw_\ell$ are linearly independent and, thus, a basis of $\mathcal{C}(AB)$.

If $Aw_1, Aw_2, \ldots, Aw_\ell$ were linearly dependent then there would exist a non-trivial linear combination

$$\begin{aligned} \alpha_1 Aw_1 + \alpha_2 Aw_2 + \cdots + \alpha_\ell Aw_\ell &= \vec{0}, \\ A(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_\ell w_\ell) &= \vec{0}. \end{aligned}$$

Notice that the second equality is a contradiction, since no nontrivial combination of the vectors $w_1, \ldots, w_\ell$ is contained in the kernel of A (that is how we chose them).