# NMAI057 – Linear algebra 1

## Tutorial 11 – with solutions

## Linear maps

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**Problem 1.** Decide and justify whether the following real functions are linear maps

(a) $f_1(x) = 0$,
(b) $f_2(x) = 1$,
(c) $f_3(x) = 2x$,
(d) $f_4(x) = x + 1$,
(e) $f_5(x) = x^2$.

**Problem 2.** Decide and justify whether the following transformations of $\mathbb{R}^2$ are linear maps

(a) $f_6((x_1, x_2)^T) = (x_1 + x_2, x_1 - x_2)^T$,
(b) $f_7((x_1, x_2)^T) = (x_1 - x_2, x_1 - x_2)^T$.

**Problem 3.** For the transformation $f_6 : \mathbb{R}^2 \to \mathbb{R}^2$ defined above, find the matrix $[f_6]_{K_2,K_2}$ of $f_6$ w.r.t. the standard basis $K_2 = \{e_1 = (1, 0)^T, e_2 = (0, 1)^T\}$ of $\mathbb{R}^2$.

**Problem 4.** Consider the basis $B_1 = \{(-1, 0, 3)^T, (2, -2, 2)^T, (0, 1, -3)^T\}$ of $\mathbb{R}^3$. Find the matrix of $f : \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. the basis $B_1$ (i.e., $[f]_{B_1,B_1}$) if you know that $f$ maps the basis vectors as follows (note that all vectors are scaled by a factor of 2):

$$f((-1, 0, 3)^T) = (-2, 0, 6)^T,$$
$$f((2, -2, 2)^T) = (4, -4, 4)^T,$$
$$f((0, 1, -3)^T) = (0, 2, -6)^T.$$

For $x$ with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the matrix $[f]_{B_1,B_1}$ to compute the coordinates $[f(x)]_{B_1}$ of the image of $x$ under $f$ w.r.t. $B_1$.

**Problem 5.** For the linear map $f$ from the previous problem, find the matrix $[f]_{B_1,B_2}$ of $f$ w.r.t. the bases

$$B_1 = \{x_1 = (-1, 0, 3)^T, x_2 = (2, -2, 2)^T, x_3 = (0, 1, -3)^T\} \text{ and}$$
$$B_2 = \{y_1 = (-1, 1, 0)^T, y_2 = (0, 1, -1)^T, y_3 = (1, 0, 1)^T\}.$$

For $x$ with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the matrix $[f]_{B_1,B_2}$ to compute the coordinates $[f(x)]_{B_2}$ of the image of $x$ under $f$ w.r.t. $B_2$.

**Problem 6.** For the bases $B_1$ and $B_2$ from the previous problem, find the change of basis matrix $[id]_{B_1,B_2}$ that transforms coordinates w.r.t. $B_1$ into coordinates w.r.t. $B_2$.

For $x$ with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the change of basis matrix $[id]_{B_1,B_2}$ to compute the coordinates $[x]_{B_2}$ of $x$ w.r.t. $B_2$.

**Problem 7.** How about transforming the coordinates $[x]_{B_2}$ of $x$ w.r.t. $B_2$ into coordinates w.r.t. $B_1$? Find the change of basis matrix $[id]_{B_2,B_1}$ that transforms coordinates w.r.t. $B_2$ into coordinates w.r.t. $B_1$.

For $x$ with coordinates $[x]_{B_2} = (1, -6, 4)^T$, use the matrix $[id]_{B_2,B_1}$ to compute the coordinates $[x]_{B_1}$ of $x$ w.r.t. $B_1$.

**Problem 8.** Consider $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5^3$ defined by the matrix

$$[f]_{B,K_3} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 4 & 0 & 3 \end{pmatrix}$$

w.r.t. the standard basis $K_3$ of $\mathbb{Z}_5^3$ and the basis $B = \{(3,2,1)^T, (1,3,4)^T, (2,2,2)^T\}$ of $\mathbb{Z}_5^3$.

Compute the matrix $[f]_{K_3,K_3}$ of $f$ w.r.t. to the standard basis $K_3$ of $\mathbb{Z}_5^3$.

**Problem 9.** Consider $g\colon \mathbb{Z}_7^2 \to \mathbb{Z}_7^3$ defined by the matrix

$$[g]_{K_2,K_3} = \begin{pmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 6 \end{pmatrix}$$

w.r.t. to the standard bases $K_2$ of $\mathbb{Z}_7^2$ and $K_3$ of $\mathbb{Z}_7^3$.

Compute the matrix $[g]_{B_2,B_3}$ of $g$ w.r.t. the bases $B_2 = \{(1,4)^T, (3,1)^T\}$ of $\mathbb{Z}_7^2$ and $B_3 = \{(1,1,2)^T, (1,0,3)^T, (6,0,5)^T\}$ of $\mathbb{Z}_7^3$.

**Problem 10.** Consider $h\colon \mathbb{Z}_5^2 \to \mathbb{Z}_5^3$ defined by the matrix

$$[h]_{B_2,B_3} = \begin{pmatrix} 4 & 3 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}$$

w.r.t. the bases $B_2 = \{(4,3)^T, (1,4)^T\}$ of $\mathbb{Z}_5^2$ and $B_3 = \{(1,1,1)^T, (1,4,0)^T, (4,0,1)^T\}$ of $\mathbb{Z}_5^3$.

Compute the matrix $[h]_{K_2,K_3}$ of $h$ w.r.t. the standard bases $K_2$ of $\mathbb{Z}_5^2$ and $K_3$ of $\mathbb{Z}_5^3$.

**Problem 11.** For the linear maps $f$ and $h$ defined above, compute the matrix $[f \circ h]_{K_2,K_3}$ of the composed map $f \circ h\colon \mathbb{Z}_5^2 \to \mathbb{Z}_5^3$ w.r.t. the standard bases $K_2$ of $\mathbb{Z}_5^2$ and $K_3$ of $\mathbb{Z}_5^3$.