NMAI057 – Linear algebra 1

Tutorial 10

Vector spaces

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Problem 1. The system of subsets of a set $A = \{a, b, c, d, e\}$ can be viewed as a vector space over the field $\mathbb{Z}_2$. Determine

- the zero vector $\mathbf{0}$,
- opposite vector $-\mathbf{u}$ to the vector $\mathbf{u} = \{b, d, e\}$,
- the result of linear combination $\mathbf{s} = 1 \cdot \mathbf{v} + 1 \cdot \mathbf{w} + 0 \cdot \mathbf{x} + 1 \cdot \mathbf{y}$, where $\mathbf{v} = \{a, c, d\}$, $\mathbf{w} = \{b, c\}$, $\mathbf{x} = \{a, b, d, e\}$ and $\mathbf{y} = \{b, e\}$,
- whether it is possible to obtain vector $\mathbf{z} = \{a, b, e\}$ as a linear combination of vectors $\mathbf{v}, \mathbf{w}, \mathbf{x}$ and $\mathbf{y}$.

Problem 2. Is it possible for the structure $(\{0, 1, 2, 3, 4, 5\}, \oplus, \odot)$ to form a vector space over the field $\mathbb{Z}_3$, where $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} \mod 6$ and $a \odot \mathbf{u} = a \cdot \mathbf{u} \mod 6$?

Problem 3. In the vector space $\mathbb{R}^4$ over the field $\mathbb{R}$ find the linear combination of vectors $(-5, 5, 1, -1)^T$, $(2, -5, 0, 2)^T$, $(3, 2, 0, -2)^T$ and $(2, -3, 1, 1)^T$ which does lead to vector $(-7, 12, 2, -4)^T$. Is this linear combination unique?

Problem 4. In the space $\mathbb{R}^4$ determine the coordinates $[u]_X$ with respect to the (ordered) basis $X = ((1, -3, 7, 2)^T, (3, 2, 1, -4)^T, (0, -1, 4, -3)^T, (-2, 4, -3, 0)^T)$ for vectors $u_1 = (2, 2, 9, -5)^T$, $u_2 = (-7, 2, 9, -8)^T$ a $u_3 = (4, -42, 31, 20)^T$.

Problem 5. In the space of real polynomials of degree at most four with the basis $X = (x^4 + x^3,\ x^3 + x^2,\ x^2 + x,\ x + 1,\ x^4 + 1)$ determine coordinates $[f]_X$ of the following vectors f:

(a) $f(x) = x^4 - 1$.

(b) $f(x) = x^4 + x^3 + x^2 + x + 1$.

(c) $f(x) = x^4 + x^2 + 1$.

(d) $f(x) = x^3 + x$.

Problem 6. Extend the set M to a basis of the vector space V

(a) $M = \{(1, 2, 0, 0)^T, (2, 1, 1, 3)^T, (0, 1, 0, 1)^T\}$, $V = \mathbb{R}^4$.

(b) $M = \{-x^2, x + x^2, x^3 - 1\}$, in the space V of real polynomes of degree at most three.

(c) $M = \left\{ \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}$ in the space $V = \mathbb{R}^{2\times 2}$.

Problem 7. Determine dimensions and bases of the following vector spaces.

(a) $U_1 = \mathcal{L}((4,1,0,3,4,0,0)^T, (4,3,1,0,2,3,1)^T, (4,1,4,0,3,2,4)^T,$
$(2,4,1,4,4,3,1)^T, (0,4,3,2,2,4,3)^T)$.

(b) $V_1 = \{(x_1,\dots,x_7)^T \in \mathbb{Z}_5^7 : x_1 + 3x_2 + x_3 + 2x_4 + 3x_5 + x_6 + 2x_7 = 0,$
$3x_1+4x_2+3x_3+x_4+4x_5+2x_6+4x_7 = 0,\ 2x_1+x_2+4x_3+4x_5+2x_7 = 0\}$.

(c) $U_2 = \mathcal{L}((1,2,4,2,3,1,2)^T, (2,3,4,1,2,1,3)^T, (3,4,1,1,4,1,4)^T,$
$(4,0,2,3,3,4,1)^T, (4,3,1,3,2,3,2)^T)$.

(d) $V_2 = \{(x_1,\dots,x_7)^T \in \mathbb{Z}_5^7 : x_1 + 2x_2 + x_3 + x_5 + 2x_6 + 3x_7 = 0,$
$4x_1 + 2x_2 + x_3 + 3x_4 + 2x_5 + x_7 = 0,\ x_1 + x_2 + 3x_3 + x_6 = 0\}$

Problem 8. Prove that if V is a subspace of a finitely generated space W then there exist bases X and Y, resp., of V and W, resp., such that $X \subseteq Y$.

Problem 9. Determine, whether the spaces U_i and V_i are in an inclusion. If so, find a basis of the larger one that extend a basis of the smaller one. These subspaces of $\mathbb{Z}_5^7$ are defined as follows:

(a) $U_1 = \mathcal{L}((4,1,0,3,4,0,0)^T, (4,3,1,0,2,3,1)^T, (4,1,4,0,3,2,4)^T,$
$(2,4,1,4,4,3,1)^T, (0,4,3,2,2,4,3)^T)$ $V_1 = \{(x_1,\dots,x_7)^T \in \mathbb{Z}_5^7 : x_1 +$
$3x_2 + x_3 + 2x_4 + 3x_5 + x_6 + 2x_7 = 0,$
$3x_1+4x_2+3x_3+x_4+4x_5+2x_6+4x_7 = 0,\ 2x_1+x_2+4x_3+4x_5+2x_7 = 0\}$

(b) $U_2 = \mathcal{L}((1,2,4,2,3,1,2)^T, (2,3,4,1,2,1,3)^T, (3,4,1,1,4,1,4)^T,$
$(4,0,2,3,3,4,1)^T, (4,3,1,3,2,3,2)^T)$ $V_2 = \{(x_1,\dots,x_7)^T \in \mathbb{Z}_5^7 : x_1 +$
$2x_2 + x_3 + x_5 + 2x_6 + 3x_7 = 0,$
$4x_1 + 2x_2 + x_3 + 3x_4 + 2x_5 + x_7 = 0,\ x_1 + x_2 + 3x_3 + x_6 = 0\}$

Problem 10. Prove that if V and W are finitely generated, then

$$\dim(V) + \dim(W) = \dim(V \cap W) + \dim(\mathcal{L}(V \cup W)).$$

Problem 11. Let V ve the set of all real symmetric matrices of order three with zeros on the diagonal.Show that V is a subspace of $\mathbb{R}^{3\times 3}$. Determine the dimension of V and find some basis of this subspace.

Problem 12. The coordinates of a vector u with respect to the ordered basis $X = (v_1, v_2, v_3, v_4)$ are $[u]_X = (a_1, a_2, a_3, a_4)^T$. Determine the coordinates of the same vector with respect to the basis $Y = (v_1 + v_4, v_2 + v_3, v_4, v_2)$.