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CATEGORY THEORY AND THE FOUNDATIONS OF MATHEMATICS: PHILOSOPHICAL EXCAVATIONS*

ABSTRACT. The aim of this paper is to clarify the role of category theory in the foundations of mathematics. There is a good deal of confusion surrounding this issue. A standard philosophical strategy in the face of a situation of this kind is to draw various distinctions and in this way show that the confusion rests on divergent conceptions of what the foundations of mathematics ought to be. This is the strategy adopted in the present paper. It is divided into 5 sections. We first show that already in the set theoretical framework, there are different dimensions to the expression 'foundations of'. We then explore these dimensions more thoroughly. After a very short discussion of the links between these dimensions, we move to some of the arguments presented for and against category theory in the foundational landscape. We end up on a more speculative note by examining the relationships between category theory and set theory.

The advent of category theory as a foundational framework forces us to face some difficult questions. Here is a very short sample. Should category theory be compared to group theory or set theory? What are the links between category theory and logic? Should a foundation of mathematics be normative? Should it provide guidelines for the development of mathematics? Is this role played by set theory? Are set theory and category in conflict with one another, or, in other words, is category theory an alternative to set theory?[1]

Furthermore, when we bring in topos theory into the picture, the tension becomes acute.[2] Some toposes can be considered to constitute an appropriate foundation for either all of 'ordinary' mathematics or for some portion of it, e.g. differential geometry.[3] However, since category theory cannot be carried out completely within topos theory without some adjustments which are not purely topos theoretical, one could argue that it cannot constitute a proper foundation for all of mathematics.[4] But then again, one could claim that toposes and categories are just structured sets and therefore that we are still talking about sets after all.

In the present paper, I will try to show that (1) in fact there are many different senses to the expression 'foundations of' and (2) that some of the arguments given either in favor of or against category theory are based on different conceptions of what should be included in, or what should be meant by, the foundations of mathematics. It is hoped that this will allow

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us to see precisely where the different parties disagree and, from there, orient the debate appropriately.

1. SET THEORY AND THE FOUNDATIONS OF MATHEMATICS

Let us first quickly review the traditional claim that set theory is a foundation for mathematics. Already in that claim we will see that there are, in fact, many claims involved.

As is well-known, set theory provides a conceptual unification of mathematics. There is no need to go over the technical details of this unification. The question is whether it is philosophically significant.

One answer is that this unification is an ontological reduction: we now know what mathematics is really about, namely sets. Better still, it is claimed that an intuitive picture of that universe and of its principles can be given, i.e. a *genetic* view of the universe of sets can be provided: the so-called cumulative hierarchy.[5]

This ontological reduction might very well satisfy the philosopher, for its byproducts might bring considerable philosophical payoffs: for instance, the traditional fog surrounding entities might have been lifted:

> Whatever can be said in the old-fashioned way in terms of 'abstract forms' and 'universals' can be reformulated much more precisely and simply in terms of sets, structures and formal languages. In this way we are spared the difficulty of saying just what sort of *things* those abstract forms and universals are. There is no need to conjure up some hypothetical 'entity', some *je ne sais quoi* which is the abstract form of a given structure, the *thing* which it and all its isomorphs somehow share. Set theory simply banishes the problem of universals from the foundations of rnathematics as irrelevant (Mayberry 1977, pp. 23–24).

However, the fruits of the unification cannot be cashed in until we have a clear, systematic and reasonably complete picture of this new universe.

Enter the logician who constructs a *formal* set theory and thus provides a logical foundation for mathematics. Here, the task is different: the above unification has to be reconstructed systematically. The logician has to find a proper deductive system, as well as proper primitive concepts and axioms and make sure that she has captured the above universe adequately: all or almost all of mathematics should be derivable in her formal theory. Hopefully the axioms will possess desired *epistemological* properties and, in fact, these epistemological properties might be the underlying motivation of the logician's work. For instance, they will be considered to be self-evident, or shown to be analytic or purely logical or indubitably true. They will also be heuristically valuable: simple and few in number.

Now, consider the 'working mathematician'. As far as she is concerned, the benefits of this conceptual unification does not have much to do with an ontological reduction nor a logical presentation. It means essentially two things. Firstly, the various constructions needed throughout mathematics are built up from a uniform tool kit: unions, intersections, products, quotient objects, indexing in general, forming the set of functions from a set to a (not necessarily distinct) set. Secondly, it allows for distinctions of size, which are sometimes crucial. She doesn't care whether the above are described in a first or a second or a n-order language, or whether the language allows one to distinguish between sets and classes or what-not or again whether the axioms are self-evident or simple. What she finds interesting is that set theory 'guides' her in her research, that it suggests ways of defining and constructing the appropriate objects for certain proofs in a natural manner and that it allows her to move easily from a 'small' context to a larger one. It wouldn't be inaccurate to talk about *methodological* or *pragmatic* foundations in that case.

Thus the claim that set theory is the appropriate foundation for mathematics seems to be justifiable in five radically different ways:

(1) mathematics is truly the science of the realm of sets;
(2) set theory is part of logic, the latter being the universal science upon which every other science is based; Set theory is just, in a sense, applied logic to mathematical concepts;
(3) set theory captures the fundamental, i.e. the most general, cognitive operations upon which the whole of mathematical knowledge is based;
(4) the axioms of set theory possess an epistemological property, e.g. self-evidence, truth, indubitability, which gives them a priviledged status;
(5) a set theory is indispensable for doing mathematics, if only to provide a uniform and good control on questions of size, but mostly for definitions, constructions and techniques of proofs. Thus a set theory is heuristically and methodologically inescapable.

So, we see that the expression 'foundation(s) of' has different dimensions. Firstly, we have what is usually, and incorrectly, called the *ontological* foundations of a field: the objects this field is supposedly talking about. Secondly, we have the *logical* foundations of a field: in principle, this gives us the 'logic', i.e. the theory of deduction, used in that field together with the (relative) basic concepts involved, the latter given by the axioms of the theory. Thirdly, we have a *semantical* foundation; by linking appropriately the formal system with the 'ontology', we in principle know how and to what our language refers and how and what it means. Fourthly, we have the *methodological* or *pragmatic* foundations, which show what are the principles, methods and concepts used to construct and analyze the different objects of a domain. Finally, and this one is also very important, we have an *epistemological* foundation of a field. For instance, we might have shown that a portion of mathematics is analytic and, therefore, that this portion possesses all the properties of analytic knowledge, whatever these properties might be. Let us explore these distinctions more thoroughly.

2. A BRIEF ANALAYSIS OF THE EXPRESSION 'FOUNDATIONS OF'

'Being a foundation of' is clearly a binary relation, which we will write as **Found**(S, T). One of the main claims of this paper is that the distinctions we have just made involve in fact different *relations*. Our basic assumption is that these relations, despite their differences, are all binary relations between *systems* of certain types.[6]

So, let S and T be two systems. What kind of foundational relation can exist between S and T? I claim that there are at least six possible different relations between them, depending on the type of systems S and T are. The relations are, in an arbitrary order, the following.

(i) **LogFound**(S, T): S is a (relative) *logical* foundation for T.

(ii) **CogFound**(S, T): S is a (relative) *cognitive* foundation for T.

(iii) **EpiFound**(S, T): S is an (relative) *epistemological* foundation for T.

(iv) **SemFound**(S, T): S is a (relative) *semantical* foundation for T.

(v) **OntFound**(S, T): S is an (relative) *ontological* foundation for T.

(vi) **MetFound**(S, T): S is a (relative) *methodological* or *pragmatic* foundation for T.[7]

One obvious task here is to investigate the formal properties of these systemic relations. Even though it is an important and difficult task, we will skip it since our goal is to unearth the philosophical groundwork underlying the foundational discussion. We will therefore try to clarify these relations one by one in the hope that our remarks will show, on the one hand, why and in what way some of these relations have to be distinguished, sometimes even separated, and, on the other hand, how these distinctions are relevant to foundational issues. Needless to say, our presentation will not do justice to the complexity of the situation. A satisfactory analysis will have to be done elsewhere. We hope that what follows will suffice for the remaining parts of our paper and will open new avenues in the conceptual analysis of the issue. Let us now look at these different relations in turn.

(i) S is a (relative) *logical* foundation for T. This seems at first to be straightforward. Let T be a system of mathematical objects, e.g. abelian groups, Hilbert spaces, topological spaces, sets, or a geometry. Following the standard conception of the logical foundations of a field, S would then have to be a formal, possibly axiomatized, theory for T. This means that we have to specify a 'language', that is, a syntactical system with a signature, an underlying logic, e.g. first-order logic, and, whenever possible, give a list of axioms for T. S is a systematic reconstruction of T which is supposed to show explicitly what is the deductive system involved, what are the axioms needed, in other words the primitive concepts, properties and principles of construction of a field, and, as a corollary, what are the definable objects of T within S. Thus, definability, provability and satisfaction are the three essential elements involved.

Let us try to be more precise. Let Ω denote a universe of mathematical objects, still undefined, and L a universe of logics, also undefined. Both are assumed to be systems. Informally, we should have that

$$\mathbf{LogFound} \subset \mathsf{L} \times \Omega.$$

Viewed this way, **LogFound**(S, T) is a relation between logics and mathematical systems. More precisely, S should at least be an entailment system and a satisfaction relation between S and T should be specified.[8] Thus, **LogFound** requires that we can at least determine, in the most abstract manner, what is an entailment system and what is a satisfaction relation. However, there are two major conceptual pitfalls on this road.

Firstly, for any given T there are usually many different systems S available, either all sharing the same underlying logic, e.g., the different axiomatizations of group theory, or written in different logical systems, e.g., first-order versus second-order arithmetic. In the former case, we would certainly want to say that the different presentations of the system of groups are in fact 'the same'. After all, the only difference resides in the choice of the primitive operations. The collection of all definable operations is exactly the same in both cases. Thus, there is an invariant core, an invariant theory of groups.[9] The same remarks apply for the different presentations of proof calculi for first-order logic for instance. They are different presentations of the same logic. We are here more interested in the latter than in the specific presentations. In the second case, the choice of the primitive operations which are taken to be logical yields radically different structures. We would like to have whatever is necessary to compare or relate in a systematic fashion the different logics that can be used or underline a given system T.

Secondly, we can only require that T be a mathematical system. Thus, we might not be able to characterize the collection of all mathematical systems as a whole but we can determine in each case whether a given system, or a family of such, is mathematical or not. This is rather vague and we are bound to run into questions of size and the standard problems concerning the universe of all mathematical systems.[10] We do want to preserve a certain degree of flexibility, for we want T to denote the cumulative hierarchy or the category of all (small) categories or even some other non-standard structure. For that reason, the standard set-theoretical notion of satisfaction is inadequate, for T need not be a standard set-theoretical model.

Thus, we believe that any further clarification of the relation **LogFound** should be based on the following basic desiderata:

• we should separate sharply between purely structural matters and more conventional matters. For instance, specific signatures and list of axioms for a mathematical domain should *not* be part of the logical foundation per se. We want to keep apart for the moment purely logical elements from other foundational relations. Thus, particular presentations of mathematical domains should belong primarily to the cognitive and epistemological levels;

• the logical structure of a mathematical system is fixed once one has *chosen* what counts as a logical operation. The choice is not determined by purely logical considerations; cognitive, epistemological, heuristic and pragmatic considerations play a role in this choice;[11]

• however, one should be able to compare and contrast the different logics which follow from these choices.

Steps in this direction have been taken by Meseguer's.[12] According to Meseguer, a *logic* is a 5-tuple L = (Sign, sen, Mod, $\vdash$, $\models$) such that:

(i) (Sign, sen, $\vdash$) is an entailment system;

(ii) (Sign, sen, Mod, $\models$) is a conceptual system;[13]

(iii) these two systems are linked by the following soundness condition: for any $\sum \in$ Sign, $\Gamma \subset$ Sen($\sum$) and $\phi \in$ sen($\sum$),

$$\Gamma \vdash \phi \Rightarrow \Gamma \models \phi;$$

where Sign is the category of signatures, sen is a functor from Sign to Set, assigning to each signature a set of sentences, $\vdash$ is an entailment relation, Mod is a contravariant functor from Sign to the category of all (small) categories Cat and, finally, $\models$ is a satisfaction relation.

Even though this definition allows the comparison of logics with one another, since we are now in a position to define a category of logics, and in particular logical foundations with one another, it does not satisfy all our desiderata above. Firstly, it relies heavily on the notion of signature, which is purely linguistic and to a large extent conventional. Secondly, recent work in categorical logic suggests that the appropriate categorical approach to logic is at the very least 2-categorical.[14] If this is correct then a general characterization of logics is seriously compromised, since it is then hard to see why one would stop at 2 or n or what have you.[15]

A different approach satisfying all our desiderata is offered by categorical logic and the basic notion of a sketch: given a category C of mathematical structures, is C equivalent to the category of models of a sketch S? The latter notion allows us to stay away from linguistic and conventional considerations, since sketches are geometric structures, more precisely graphs.[16] A sketch is the natural syntax of a category of models and correspond to the traditional notion of a 'language'. In fact, it is possible to show that sketchability and axiomatizability within an infinitary logic $L_{\infty,\infty}$ are equivalent.

The moral here is that we still don't have a satisfactory abstract analysis of the relation of being a logical foundation of. However, we believe that we have a much better idea of what constraints such a relation should satisfy.

(ii) S is a (relative) *cognitive* foundation for T. This, in turn, can mean either of three different things. It has a strong and a weak cognitive interpretation and it has a transcendental interpretation. Let us briefly examine them.

According to the strong cognitive interpretation, one cannot know or understand T without possessing S. Thus, S provides the answer to the question: what are the operations or faculties present in the acquisition and/or the understanding of T? In this case, S ranges over mental faculties and T over mathematical systems.

In its strongest form, this is exemplified by Piaget's stages,[17] but most of the contemporary discussion concerning mathematical intuition and perception of abstract objects and their ontological impact obviously belong here.[18] Note that we are here talking about the knowing subject. The proper place of cognitive foundations in the foundations of mathematics is one of the basic points of disagreement among philosophers.[19] In particular, its links to the logical foundations always raise passionate debates. Even though this relation has not explicitly entered the debate we are examining here, it is obvious that it is in the background of various philosophical positions.

The weak cognitive interpretation can be split into two: the 'pedagogical foundation' of a field and the 'heuristic foundation' of a field. Underlying the pedagogical foundation is the question: how should one learn (or teach) T? What is required? For example, before someone starts investigating geometry or analysis in $\mathbf{R^n}$, she should take a look at $\mathbf{R^3}$ or

$\mathbf{R}^2$. Similarly, before doing complex analysis, one ought to start with real analysis. Another example is that to learn some differential geometry, one has to know some linear algebra.

We are here talking about mathematical systems: S ranges over mathematical systems. T is in this case either (i) a generalization of S, as in the case of analysis in $\mathbf{R}^n$ and $\mathbf{R}^2$; (ii) an abstraction of S, as in the case of Boolean algebras and algebras of sets; (iii) an extension of S, as in the case of the rational numbers and the integers; (iv) a theory built upon the objects and relations of S, as in the case of geometry and linear algebra or algebraic geometry and commutative ring theory.[20] The claim here is that it is easier, even mandatory, for the knowing subject to understand S before attacking T. One would like to show that this follows from the previous relation: one ought to study analysis in $\mathbf{R}^2$ before $\mathbf{R}^n$ because our cognitive make-up is such that we can actually 'see' things directly in $\mathbf{R}^2$. This is clear for (i) and (ii) above, but controversial for (iii) and (iv). Thus this second relation depends directly on the previous one. However the precise nature of this dependence remains to be elucidated.[21]

The heuristic foundation makes sense only if there is a field of mathematics which is such that it is conducive to the discovery and understanding of new facts. Set theory might be considered to be an appropriate *heuristic* foundation for (some parts of) mathematical *research*. The simplicity of the language and concepts are conducive to intuitive guessing.[22] On the other hand, the so-called 'categorical dogmas' – for instance "look for adjoints", can also lead to better understanding and new discoveries. This is where questions related to specific representations and representational systems enter the scene. It is at this level that the sketches we have used for the relation of logical foundation seems to be less adequate that the standard presentation of a theory. Syntax in mathematics is not a trivial matter: understanding, discovering and proving new results are sometimes made possible by the introduction of an adequate syntax. Clearly, a given representational system is more appropriate than another one *for us*; it suggests certain directions, generalizations and abstractions. Again, this is where set theory is so appealing: every 'ordinary' mathematical object can be encoded in set theory.[23] The question is how these seemingly 'psychological' factors turn out to be relevant and sometimes crucial in the discovery of new facts, even in the applications of mathematics.[24] Again, this relation should be linked and even subsumed to the first relation of cognitive foundations.

As is well-known, the cognitive foundations of a field can be the reverse of the logical relation between the two fields. For instance, the logical foundations of a conceptual system might be cognitively less accessible than the concepts it purports to found.[25]

We now come to the transcendental interpretations. These are of course the a priori faculties or conditions which we should necessarily possess or fulfill for a type of knowledge to be possible. These faculties or conditions should of course be given by a transcendental method of some sort.[26] It is the latter which distinguishes this relation from the first one.

(iii) S can be a (relative) *epistemological* foundation for T. Epistemological foundations are associated with a desired epistemological property, e.g. analyticity, certainty, rationality, objectivity, self-evidence or what have you. Given one of these properties, e.g. certainty, the epistemological foundation is supposed to explain why or how T possesses this property. For instance, a logicist might have said that logic provides the epistemological foundation of mathematics.[27]

This explanation can exclude psychological or cognitive elements, which is why we have to separate it, at least in principle, from the cognitive foundations. In general, the search for epistemological foundations constitutes a foundationalist enterprise. This is an additional important difference with the cognitive foundations.

Thus, the general strategy is as follows: S should denote a body of knowledge which is chosen for its epistemological properties. Then T denotes a body of knowledge which is linked to S in such a way that the desired epistemological properties of S are transmitted to T. The simplest way to assure this transmission is usually via a reduction of T to S. But this is only one way. For instance, if T is a conservative extension of S, then we also have an appropriate link. Another example is given by interpretations of one theory into another. Troublesome notions or propositions are sometimes justified by means of interpretations in terms of accepted notions or propositions. Non-euclidean geometries gained respectability when interpretations in terms of Euclidean geometry were given. This shows that the epistemological foundations denote a collection of relations. It remains to find a way to capture abstractly the common properties of these relations.

(iv) S can be a (relative) *semantical* foundation for T. This seems straightforward at first: there are numerous examples in the history of mathematics of axiomatic systems which had no models and thus were without any semantical foundations for a while. Suffice it to mention the development of non-euclidean geometries in the 19th century. Hence, S is here a 'universe' of objects with adequate properties, a model in the usual sense of that expression. Thus, at first sight, it seems that the relation of semantical foundations is the converse of the relation of logical foundations: **SemFound**(S, T) = **LogFound**(T, S)^{op}. This should not be surprising, for, since Tarski, a semantics, that is a theory of reference and truth, is based on a satisfaction relation.

However, as in the case of the relation of logical foundations, as soon as we try to be more exact, the situation becomes more delicate. The problems are in a sense the duals of those we have expounded in the section on the logical foundations. However, they may not be solvable in the same way and at the same time. For instance, it is far from clear that we can and should ignore questions related to specific presentations and languages in the semantic foundations. Since the basic problem here is the question of the notion of reference, we might need to be very sensitive to linguistic issues, which we believe should be avoided as much as possible in the purely logical analysis. Unfortunately, these considerations would take us away from our main concern and will simply be ignored here.

(v) S can be an (relative) *ontological* foundation for T. What we have in mind here is an answer to the question: what are the entities of T 'made of'? What kind of existence do they possess? Notice that T can be any type of system whatsoever. When T is a formal system, then one is tempted to think that the ontological relation is the projection of the reference relation, hence of the semantical relation. However, this is a mistake, for here we would be interested on the ontological status of the formal system *qua* formal system and not as a system denoting something else.

Of course, since our discussion is oriented towards the foundations of mathematics, we are here more concerned with the ontological status of mathematical entities and thus T should be a system of mathematical entities. In this case, S can be either a world of transempirical and extramental mathematical entities, or a system of concepts, e.g. mental processes, or again empirical entities or processes of some sort. Here, perhaps surprisingly, we claim that the knowing subject is right at the center of the picture. In fact, we believe that it is impossible to separate the ontological foundations from the cognitive foundations. Indeed, we claim that in this case, T should be the system of cognitive functions involved in a mathematically active subject. Thus, in this case, S tells us what is ontologically underlying a mathematical activity regarded as a cognitive process. In a way, this ontological relation and the first cognitive relation are two faces of the same coin. Even though they can be distinguished, they cannot be separated.[28]

(vi) S is a (relative) *methodological* foundation for T. In this case, the question we are trying to answer is: what are the principles or methods which guarantee that an object with a given property is legitimate, that it can be or that it is different from the others of the same type? Thus the concepts of S are used as tools either to 'create' or classify or prove some facts about the objects of T. Mathematicians now commonly use the expression 'foundations' in this sense. Almost every book in commutative algebra or commutative ring theory opens up with a claim that "in addition to being a beautiful and deep theory in its own right, commutative ring theory is important as a foundation for algebraic geometry and complex analytic geometry"[29] (Matsumura 1986, p. ix). It is also illustrated by the use of group theory in topology.

I want to emphasize the fact that this contitutes a new usage of the expression 'foundations of'. For, by bringing in these tools in a given field, we are *not* providing ontological foundations *nor* logical foundations *nor* any foundation in the previous senses given. Groups now play a crucial role in topology and geometry. In fact, the whole of algebraic topology rests on the interrelations between topological spaces and groups. But these interrelations do not modify the notion of a topological space, nor do they provide us with a new logical foundation for topology, nor do they tell us how to interpret a theory in topological spaces, etc. It is a genuinely new relation between two mathematical systems. S and T are generally cognitively and logically independent of one another. In fact, in most cases the objects of S are constructed from the objects of T, as in the case of groups in topology. Thus they appear as properties of the original objects written or presented in a different language.

Notice also that a methodological foundation can play an important normative role in a given domain: it gives us the tools to look for new objects and properties. It is not clear that any of the other types of foundations can play the same role. In fact, the logical foundations are in a sense constructed from the methodological foundations. The methodological foundations appear during the construction of a field, whereas the logical foundations constitute a reconstruction of the given field from a specific standpoint.

3. LINKS BETWEEN THE RELATIONS

Let us quickly settle an obvious question: what is the relationship between the different foundational relations presented above and philosophy of mathematics? A philosophy of mathematics amounts to an ordering of the above relations. Thus, within a philosophy of mathematics, some of these relations lose their foundational status, since they are presumably shown to follow from one or a few others, and some are ignored altogether. In particular, the debate surrounding the foundational status of category theory rests on the fact that the parties involved order these relations differently, as we will try to show in the next section.

4. CATEGORY THEORY IN THE FOUNDATIONAL LANDSCAPE: IN AND OUT

The different positions and the 'debate' concerning the place and status of category theory in the foundations of mathematics is not entirely unlike the Russell-Poincaré debate concerning the place and status of logic in the foundations of mathematics. As I will try to show, those who believe that category theory should occupy the center of the foundational stage and force set theory to resign as chairman of the board are picking out certain senses of the expression 'being a foundation of' and consider the others as being irrelevant, whereas those who are against category theory as chairman believe these others senses are essential and help to show that category theory cannot fulfill the requirements for the position. Thus, basically, each camp is working with a different list of the senses or roles that should be covered by foundations.

4.1. *IN*

Let me first present the argument for the view that category theory is the appopriate foundation for mathematics. Let me start with some examples.

In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are *those which can be stated in terms of their abstract structure rather than in terms of the elements with the objects were thought to be made of.* The question thus naturally answers whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, ... *Here by "foundation" we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved*[30] (Lawvere 1966, p. 1) [emphasis mine].

Lawvere is here endorsing the traditional view of the logical foundations of mathematics: an axiomatic presentation of the category of categories in a first-order language. Notice the steps involved here: first, the recognition that category theory unifies mathematics in a certain manner; second, the conviction that this conceptual unification can be translated into an ontological reduction and, third, the attempt to achieve this reduction by building an adequate logical foundation.

But in 1969, Lawvere presents a different picture:

Foundations will mean here the study of what is universal in mathematics. Thus Foundations in this sense cannot be identified with any "starting-point" or "justification" for mathematics, though partial results in this direction may be among its fruits. But among the other fruits of Foundations so defined would presumably be guide-lines for passing from one branch of mathematics to another and for gauging to some extent which directions of research are likely to be relevant (Lawvere 1969, p. 281) [emphasis mine].

This sense of the expression 'foundations' is certainly not incompatible with the previous one. However, it pushes aside one traditional foundational relation, namely, that it should provide a justification of some sort for mathematics. In this foundational picture, one type of epistemological foundation is absent. With it disappears the logical relation also, the so-called "starting-point". We are clearly moving, in our terminology, to an autonomous *methodological* foundation together with a *heuristic* spin-off. The claim here is that category theory allows you to see and understand what makes certain constructions and results possible, in the same way that physics, say, makes you understand why buildings can stand up and others cannot. Indeed, the universals Lawvere has in mind here are the universal arrows of category theory.[31]

However, also in 69, Lawvere, together with Tierney, isolated the notion of an elementary topos which allowed him and others to develop a modified versions of his 66 views. One of these versions is expounded by Lambek and Scott (1986), where we read:

> We believe that type theory is the proper foundation for mathematics (Lambek and Scott 1986, p. viii).

But every type theory gives rise to a topos and, in particular, pure type theory generates what is called the free-topos, what the authors believe to be the universe of mathematics.[32] Interestingly enough, we find also the following:

> We are tempted to follow Lawvere and adopt the view that the growth of mathematics should be guided by various categorical slogans or the more widely held view that category theory underlines the general principles common to different areas of mathematics (Lambek and Scott 1986, p. 126).

We have here a complete separation of the logical and the methodological dimensions.

Many authors, however, claim that the truly foundational relation is the methodological foundation. Thus, Hatcher declares:

> It appears more and more clearly that what is truly foundational is not some arbitrary [sic!] starting point (some list of axioms for a comprehensive system or other), but certain key, unifying notions common to many different aspects of mathematical practice.... The notions of universality and naturality in category theory are clearly just as important [as the comprehension scheme] ... (Hatcher 1982, p. 312).

What is fascinating about the key notions of category theory, e.g. natural transformations, adjointness, etc., is that they cannot be taken as primitive. Indeed, categories were invented by Eilenberg and Mac Lane in order to define the notion of natural transformation. So we are in a position where what seems to be fundamental, e.g. adjointness, is *not* primitive from a logical point of view. However, it is clear that we *are* dealing with something fundamental.[33]

We can now give a list which sums up the major claims found in the field:

(1) category theory is heuristically fundamental;[34]
(2) the theory of the category of all categories is the ontologico-logical foundation for mathematics;
(3) category theory provides a methodological foundation for mathematics.[35]
(4) toposes provide an adequate logical foundation for 'ordinary' mathematics via their internal language; some have suggested that the free topos should be taken as the foundation for mathematics, e.g. Lambek and Scott, whereas others that a theory of well- pointed toposes with choice should be investigated, e.g. Mac Lane;
(5) topos theory provides the appropriate framework for the investigation of 'local' logical foundations, that is foundations for specific parts of mathematics, e.g. differential geometry or algebraic geometry; moreover, the axioms for a topos constitute the foundational invariants of the logical foundations of mathematics (see Bell (1988) for more on this).

Notice that this list is not consistent nor is it meant to be. Its sole purpose is to show that only specific foundational relations are mentioned: ontological, logical, methodological and heuristic. Two relations are remarkably absent: the first cognitive relation and any epistemological relation. I will now try to show that the arguments presented against category theory rely precisely on these relations.[36]

4.2. *OUT*

The best case against category theory has been presented by Feferman and Bell.[37] Feferman (1977) and Bell (1981) present similar arguments. Let us see what their target is.

> In what sense could category theory serve as a foundation for mathematics? There seems to be (at least) two possible senses: first, a strong sense, in which *all* mathematical concepts, *including* those of the current logico-mathematical framework for mathematics, are explicable in category-theoretic terms. And secondly, a weaker sense in which one only requires category theory to serve as a (possibly superior) substitute for axiomatic set theory in its present foundational role (Bell 1981, p. 353).

Take the second sense first. Both Feferman and Bell consider the work by Cole (1973), Mitchell (1972) and Osius (1974) which provides an axiomatization for well-pointed toposes in categorical terms and show that the resulting theory is equivalent to 'bounded' Zermelo set theory. Feferman then argues as follows: toposes are just a special type of categories. Thus if we can show that categories in general do not have a foundational role, *a fortiori*, we will have shown that toposes do not either. He then proceeds to give such an argument, to which we will come in due course. Therefore the above work on toposes does not show that toposes constitute a substitute to axiomatic set theory.

Furthermore, Bell objects that the translation is awkward and has "a factitious character which renders it unsuitable as a means of formalizing those mathematical notions which are normally expressed set-theoretically" (Bell 1981, p. 355). But presumably one would not work with the translation but with the topos axioms directly.

A different but related objection is that the axioms for well-pointed toposes are too weak. The absence of an equivalent form of the axiom of replacement does not allow to capture the full strength of ZFC. However, it is possible to mimic the construction of the cumulative hierarchy in such toposes.[38] Furthermore, if certain toposes were provably equiconsistent with ZFC, then they would simply be uninteresting, for then the choice between the two would be a matter of taste and not a mathematical issue. The advantage of the topos theoretic framework is that it allows a systematic investigation of the relationships between toposes, i.e. between alternative foundations for mathematics, something which cannot be done so easily and with the same guiding principles in a set theoretical framework.

Let us move to the more general argument given by Feferman:

> The point is simply that *when explaining* the general notion of structure and of particular kinds of structures such as groups, rings, categories, etc. we implicitly *presume as understood* the ideas of *operation* and *collection* (Feferman 1977, p. 150).

In other words, the notion of class and of operation are prior, clearly in a cognitive sense, to any structural notions such as those used in category theory. The crucial assumption here seems to be that a logical foundation for mathematics – which according to this argument should be a theory based on the notions of collection and operation – has to be expressed in terms of the concepts which are cognitively fundamental.[39] For if the argument merely attempts to establish that the theory, whatever it is, has to be taught before category theory, then it is simply irrelevant. Indeed, assuming for the sake of the argument that the latter is true, it is still possible for category theory to be more adequate as a logical foundation. Thus, the argument has to rely on a strong link between the cognitive foundations and the logical foundations *and* the claim that the notion of class and operation are cognitively fundamental. What is the status of this last claim? Either it is an empirical claim and therefore ought to be tested accordingly or it is some sort of transcendental argument asserting that any notion of structure is necessarily based on the notions of collection and operation.[40] In the former case, research in the foundations of mathematics would then rely on experimental results and in the latter case, one has to clarify the type of necessity involved, a thorny issue if there is one.

However, two additional facts have to be pointed out. Firstly, no one has so far claimed that category theory ought to be the logical foundations of mathematics because it reflects basic cognitive faculties. In other words, the link between the cognitive and the logical is absent from the categorical picture. As we have seen, the claim is either ontological or methodological, but never cognitive nor epistemological. Hence, the different 'sides' are organizing the relations in incompatible ways. There is clearly a profound disagreement about the very nature and functions of the foundations of mathematics. Secondly, the above argument could be turned on its head and become an argument *in favor* of category theory. This can be seen in two ways. On the one hand, it is simply not true that any structural notion has to be defined via the notions of collection and operation. The latter approach is a *dynamical* approach to structures, and it is certainly a legitimate one. However, there is a *static* or *geometric* approach to structures which is based on the notion of collection, which is inevitable, and the notion of relation, not operation. Groups, rings and all algebraic structures including categories can certainly be looked at from the dynamical point of view. But there are other types of structures, geometrical and topological, among which we *can* include categories, which are better seen as static or geometric objects and this point of view is even more perspicuous, even unavoidable, when one considers higher-dimensional category theory.[41] Thus one can *explain* categories directly in terms of geometry, thus avoiding the notion of operation. On the other hand, even within the dynamical point of view, one can claim that the basic operations of category theory are precisely the notions of collection and operation, in this case of morphism. Hence, it is possible to use Feferman's argument to defend the claim that category theory is precisely the theory which captures these notions at the right level of generality. This latter claim would link the logical foundations with some sort of general cognitive element.

5. SETS AND CATEGORIES AS ENCODINGS

We have on our hands two conceptual systems which unify mathematics in different manners. Moreover, the relationships between these two systems have yet to be fully clarified (see Blass (1984) for a review). From a conceptual point of view, the following relationships are possible:

(i) categories are structured sets; thus category theory is a special chapter of set theory;
(ii) sets are unstructured categories; thus set theory is a special chapter of category theory;[42]
(iii) set theory and category theory are complementary ways of organizing the mathematical universe, irreducible to one another; any mathematical system can be represented as a set or as a category, depending on the context and the needs; this complementarity is the traditional complementarity between arithmetic and geometry and cannot be avoided, since it reflects basic features of our cognitive make up;[43]
(iv) sets or categories (or both) will simply disappear since they are merely convenient notational systems, in the same way that the concept of ether disappeared from physics.[44]

Clearly, (i) and (ii) are motivated by the belief that the reducing entities constitute the fabric of the mathematical universe. Hence, underlying these positions we find strong and conflicting ontological convictions. However, both claims face technical difficulties. (i) is acceptable if and only if we can settle the question of the set theoretical foundations of category theory. Indeed, developing category theory within set theory is not a trivial exercise. The major problem is the incessant need for 'large' categories. One has to make either semantical or syntactical adjustments. The semantical adjustment consists to take refuge in Grothendieck universes, which amounts to the acceptance of the existence of innacessible cardinals. The syntactical solution is to use reflection principles, which is equivalent to the claim that the universe of small sets is an elementary substructure of the universe of all sets (see Feferman (1969) for the latter). Even though both solutions are technically sound, more work has to be done before we can settle for one or the other or abandon them altogether. Similarly, (ii) is acceptable if and only if one can show that set theory can be discarded without any serious loss in all fields of mathematics, in particular in category theory itself. Again, only more research will allow us to draw the proper conclusion.

(iii) and (iv) are of a different nature altogether. They are based on the belief that there are no final and all-encompassing logical (and semantical) foundations for mathematics. In (iii), set theory and category theory are taken to be representational systems, both having their virtues and defects. They both capture essential aspects of mathematics in different ways. In particular, their interaction should yield interesting dividends. (iv) is sheer conjecture.

6. CONCLUSION

It seems to us that at this stage there is no a priori argument which can justify the exclusion of either set theory or category theory from the foundations of mathematics. Category theory has at the very least an important methodological role to play in contemporary mathematics, in the same way that set theory played a crucial methodological role during the late 19th and this century. It is tempting to conclude that this is all one can ask from a foundational framework. But this attitude ignores the cognitive and epistemological dimensions that are inseparable from foundational work. Foundational research is often motivated by purely philosophical considerations, which can be remote from current mathematical practice. It is these motivations that have to be compared and contrasted. They should promote investigations going in different directions, not curb alternatives.

NOTES

* Various versions of this paper have been read by many people, many of whom have made crucial comments. Needless to say, I am entirely responsible for the claims made in this paper. I would particularly like to thank, in alphabetical order, Mario Bunge, Marta Bunge, Michael Hallett, Andrew Irvine, Saunders Mac Lane, Collin McLarty, Peneloppe Maddy and Mihaly Makkai. Part of the work was done while the author was a visiting fellow at REHSEIS in Paris and at the Center for Philosophy of Science in Pittsburgh. I would like to thank everyone for his or her help and support. I gratefully acknowledge the financial support received from the SSHRC of Canada while this work was done.

[1] Contradictory claims abound in the literature. Here is a sample of divergent claims on various issues.

(i) The normative character of category theory:

The subject Category Theory, ..., neither is a normative subject, nor ... (Kuyk 1977, p. 162).

We are tempted to follow Lawvere and adopt the view that the growth of mathematics should be guided by various categorical slogans ... (Lambek and Scott 1986, p. 126).

(ii) Category theory as a general tool:

Category theory has long passed the confines of a "theory" and truly is one of the fundamental mathematical tools, unifying many disciplines much the way group theory does (Faith 1981, p. ix).

In the light of complementarism Category Theory should be rather viewed as an interdisciplinary important language device creating mathematical thinking economy, than as a discipline comparable to, say, Group Theory as it rose in the middle of the nineteenth century (Kuyk 1977, p. 163).

(iii) The normative character of set theory:

... set theory regulates the usage of the mathematical language as it bears on the formation of new totalities of entities from given totalities. Thus, set theory obtains a *normative* character pertaining to the mathematical language (Kuyk 1977, p. 147) (emphasis added)

The function of set theory in the foundations of mathematics is a logical one... because it is concerned with the logical foundations, rather than with the organization of mathematics, it does not have anything at all to say concerning what definitions *ought* to be made, or which structures, among the *a priori* possible ones, might prove to be of mathematical interest (Mayberry 1977, p. 18).

(iv) The foundations of mathematics and the abstract nature of mathematics:

It is obvious from everything I have just said that the proposal made by certain category theorists to replace set theory by a foundational theory which more adequately reflects the 'abstract' nature of modern mathematics is completely misguided (Mayberry 1977, p. 25).

In two senses, set theory is not sufficiently abstract to serve as foundations of mathematics. It might be said that we have real numbers as a basic datum, and it is less central how reasoning about real numbers is formalized. In another direction, mathematics is interested in abstract structures such as groups and fields, though involving concepts like that of set, are independent of the detailed structures of our set theory (Wang 1974, 1983, p. 554).

[2]For a presentation of some of the confusion surrounding topos theory, see McLarty (1990).

[3] See Moerdijk and Reyes (1991).

[4]This is in itself a complex issue which would require a whole paper. The fundamental problem is the same as when one tries to develop category theory within a set theory: how to deal with large categories. Some tinkering has to be done to allow the latter. Various strategies are possible: Grothendieck universes, reflection principles, proper classes, fibered categories, to mention only the best known. See Mac Lane (1972) or Makkai and Paré (1989) for the former, Feferman (1969) for reflection principles, Chapman and Rowbottom (1992) for proper classes in topos theory and Bénabou (1985) for fibered categories. I thank both referees for their comments on this point.

[5]See Wang (1974) or Shoenfield (1977) for a technical presentation; for a critical examination of the genetic approach, see Boolos (1971), Parsons (1983) and Hallett (1984).

[6]Pick your favorite characterization of a system. I would adopt Bunge's definition. See Bunge 1979 for a formal definition. However, since no specific details of that definition play any role here, we will simply skip it.

[7]We are ignoring whether a temporal dimension should be attached to these relations. The question is far from trivial. For instance, Frege and the early Russell would not have accepted any temporal element in the foundational picture. Recent writers have on the contrary insisted on the fact that since mathematics evolves through time, a foundational system will have to evolve accordingly. This immediately yields a criterion: a foundational system is inappropriate if it cannot accomodate such changes. See Bénabou (1985) and Bunge (1985) for more on the temporal aspect of the foundations of mathematics.

[8]We should point out that this is different from what people had in mind up until the

beginning of the 20th century. Indeed, it seems rather clear that, until then, a logical foundation for a field consisted exclusively in an axiomatic reconstruction of the given field. The purpose of the logical foundation was to allow formal proofs from a list of axioms of fundamental results of the given field. In other words, the satisfaction relation did not play any role.

[9]The search and presentation of such an invariant theory was part of Lawvere's original motivation to apply category theory to logical issues. See Kock and Reyes (1977) for a presentation and a brief discussion.

[10]Don't ask, it is now too easy... Yes, the universe of all mathematical systems ought to be a mathematical system itself and all the rest. But we have *not* assumed that we are working in sets and that T should denote an element of a universe of sets.

[11]This is especially clear in categorical logic. A given category possesses all kinds of limits, colimits and exactness properties. Some of these have to be selected or identified as logical operations.

[12]See Meseguer (1987) for details. To be fair, we should point out that Meseguer's main contribution in his paper in his clarification of the notion of a proof calculi and its relations to the other notions.

[13]Meseguer, following Goguen and Burstall (1984, 1986), call such a system an institution.

[14]See Makkai and Paré (1989) for instance.

[15]M. Makkai, personal communication.

[16]"Sketches are not designed as notations, but as a mathematical structure embodying the formal syntax" (Barr and Wells 1990, p. 172). For more on sketches, see Barr and Wells (1985, 1988, 1990) and Makai and Paré (1989) for the equivalence mentioned. Needless to say, more should be said bout sketches and their implications for the logical foundations of mathematics. Unfortunately, it would take us too far from the topic at hand and will have to be dealt with elsewhere.

[17]Interestingly enough, Piaget thought that Bourbaki's structures constituted some kind of foundation for mathematics and these structures were somehow reflected in a child's cognitive development. Of course, Bourbaki's mother structures are foundational only in an informal sense, but this is a different issue. See Corry (1992) for more.

Different from this idea of a succession of fixed stages, we find claims like those of Dedekind: "So from the time of birth, continually and in increasing measure we are led to relate things to things and thus to use the faculty of the mind on which the creation of numbers depends" (Dedekind 1901, p. 34). Or again: "If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. Upon this unique and therefore absolutely indispensable foundation, ..., must, in my judgment, the whole science of numbers be established" (Dedekind 1901, p. 32). Dedekind's claim is clear: behind the notion of numbers, we find more fundamental operations of the mind, cognitive faculties, from which numbers are created.

[18]A large part of the contemporary literature in philosophy of mathematics turn on this issue. To mention but a few: (Kitcher 1983; Tieszen 1989; Maddy 1990; Field 1989; Parsons 1983, 1990; Resnik 1981, 1982, 1988).

[19]Some, like Frege and Russell, claim that it simply does not belong to the picture at all. But even they had to leave room for some sort of perception of abstract objects. "The discussion of indefinables – which forms the chief part of philosophical logic – is the endeavor to see clearly, and to make others see clearly, the entities concerned, in order that the mind may have that kind of acquaintance with them which it has with redness or the taste of a

pineapple" (Russell 1903, p. xv). Russell goes on to declare that "I have failed to perceive any concept fulfilling the conditions requisite for the notion of a class" (Russell 1903, pp. xv–xvi). But this faculty seems to be simply undefinable for Russell and its properties have no function in the foundational picture.

Others, following Poincaré, want to include these faculties in the foundations of mathematics: "M. Russell me dira sans doute qu'il ne s'agit pas de psychologie, mais de logique et d'épistemologie; et moi, je serai conduit à répondre qu'il n'y a pas de logique et d'épistemologie indépendantes de la psychologie" (Poincaré 1909, 1986; p. 482). See also Goldfarb 1988 for more on this debate.

[20]We believe that these four relations are in general different and have to be distinguished. Of course, every abstraction is a generalization, but the converse is not true, e.g. the generalization from $\mathbf{R}$ to $\mathbf{R}^2$. In fact, in some cases a specialization might be more abstract than the starting point, e.g. flatlanders vs. 'ordinary' three-dimensional worlds. (iii) and (iv) sometimes collapse: the reals can be thought of as an extension of the rationals or as built out of the rationals.

[21]We find a beautiful example of this dependence in De Morgan's writings: "Geometrical reasoning and arithmetical process have each its own office; to mix the two in elementary instruction, is injurious to the proper acquisition of both" (De Morgan, quoted by Mathias 1992, p. 10).

[22]"All branches of mathematics are developed, consciously or unconsciously, in set theory or in some part of it. This gives the mathematician a very handy apparatus right from the beginning. The most he usually has to do in order to have his basic language ready is to describe the set theoretical notation he uses" (Levy 1979, p. 3). This claim is certainly hard to falsify . . . I suppose we could also claim that mathematicians were doing set theory "unconsciously" all along.

[23]"Does set theory have some essential structural property that guarantees its ability to encode other theories? Does set theory serve as a foundation for merely those theories that have been constructed in the past, with no expectation that it will serve for future theories? Or is there something about human brains that prevents them from producing mathematics that cannot be coded in set theory? My guess is that the historical view is closest to the truth, but for psychological reasons; mathematics codable into set theory was produced first (and we have not progressed beyond it) because it is easier for our minds to grasp" (Blass 1984, p. 26).

[24]Steiner mentions matrix representations and their uses in the discovery of the eight-fold way. See Steiner (1989).

[25]This was recognized by Russell, among others: "There is an apparent absurdity in proceeding, as one does in the logical theory of arithmetic, though many rather recondite propositions of symbolic logic, to the "proof" of such truisms as $2 + 2 = 4$: for it is plain that the conclusion is more certain than the premises, and the supposed proof therefore seems futile" (Russell 1907, p. 272).

[26]Examples abound. Consider first the following passage from Hilbert: "No more than any other science can mathematics be founded by logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us on our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, *it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction*. This is the

basic philosophical position that I regard as as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. *This is the very least that must be presupposed: no scientific thinker can dispense with it, and therefore everyone must maintain it, consciously or not*" (Hilbert 1927, 1967, p. 465) (my emphasis).

Consider this now: "What we must begin with, in this domain, is the possibility of distinguishing various points form one another. This may be designated, with Veronese, as the first axiom of Geometry. How we are to define a point, and how we distinguish it from other points, is for the moment irrelevant" (Russell 1897, 1956. p. 119).

It is not clear whether we have to be able to distinguish various points perceptually or rather conceptually, i.e. by being able to name different things called "points". Be that as it may, Russell attempts to found the whole of (projective) geometry by a transcendental deduction: " ... I wish to point out that projective Geometry is wholly *a priori*; that it deals with an object whose properties are logically deduced from its definition, not empirically discovered from data; that its definition, again, *is founded on the possibility of experiencing diversity in relation, or multiplicity in unity; and that our whole science, therefore, is logically implied in, and deducible from, the possibility of such experience*" (Russell 1897, 1956, p. 146) (my emphasis.)

[27]In fact, this is the whole point of logicism. The property Frege attributed to logic and hoped to spill over mathematics – minus geometry – was analyticity. Russell's case is more delicate. It seems that his main objective was to get rid of Kant's form of intuition in mathematics. In the *Principles of Mathematics*, he was forced to adopt some sort of direct abstract perception and Moore's awkward ontology. In fact, during this period, Russell seems so eager to avoid any contact with psychologism that the epistemological dimension is almost absent in his work. This changes as he tries to link logic with epistemology. (See Hylton (1980, 1990) and Irvine (1989) for more on Russell's position.)

[28]As we have already mentioned, this link is at the center of contemporary philosophy of mathematics. See note 18 for references.

[29]An anonymous referee pointed out to me that Griffiths and Harris 1978 chapter 0 is named "Foundations" and goes from several complex variables, analytic and algebraic varieties to Kahler manifolds.

[30]If Lawvere had succeeded in presenting the foundation of mathematics in this way, we could reasonably argue that a mutant of logicism has been brought back to life under a new name. For logicists, or at least Frege and Russell, were trying to provide a foundation for mathematics by constructing a single system of (higher-order) axioms "in which all mathematical objects can be defined and all their usual properties proved", just as set theory seems to do right now, but furthermore their early attempts were based on the belief that their axioms were about *concepts* (or *terms*) in general, not *elements* and sets, and that the objects derived from these concepts were derived in a purely logical manner. This was at least Frege's position and that of the early Russell, before he moved to a purely extensional point of view under Wittgenstein's and Ramsey's influence. To complete the connection, we would simply have to link concepts to abstract structures, something which has already been done by categorical logicians. (See Makkai and Reyes (1977). I have taken a closer look at some of the connections between logicism and category theory elsewhere. See Marquis (1993).) It would be interesting to examine the different axiomatizations of the category of all categories from the perspective of logicism.

[31]Thus, Lawvere was well aware of some of the foundational dimensions and the fact that they can be separated.

[32]See also Couture and Lambek (1991) and Lambek (1992) where it is argued that the free topos in fact allows a unification of the various traditional philosophies of mathematics.
[33]It can be argued that we are still dealing with logically fundamental aspects of mathematics. See Marquis (1993).
[34]In fact, topos theory might have to be included here also: "In saying that the future of topos theory lies in the clarification of other areas of mathematics through the application of topos-theoretic ideas, I do not wish to imply that, like Grothendieck, I view topos theory as a machine for the demolition of unsolved problems in algebraic geometry or anywhere else .... I do believe that the spreading of the topos-theoretic outlook into many areas of mathematical activity will inevitably lead to the deeper understanding of the real features of a problem which is an essential prelude to its correct solution" (Johnstone 1977, p. xvii).
[35]Here is yet another way to put it: "Foundations should provide general concepts and tools that reveal the structures and interrelations of various areas of mathematics and its applications, and that help in doing and using mathematics" (Goguen 1991, p. 67).
[36]I will not consider Mayberry's arguments here, as presented in Mayberry 1977. They seem to rely on the claim that behind any piece of formal mathematics lies informal set theory, that the latter is simply inescapable. If this is correct, then his arguments seem to me to be based on a similar claim than the arguments I will consider later in this paper. But I am not sure I understand his claims. They seem to me to be based on unorthodox views of the notion of abstract structure and even set theory. To mention only but one example, he claims that the notion of isomorphism is a set-theoretical notion. I fail to understand this. The notion of isomorphism in set theory amounts to a one-to-one onto mapping and I fail to see how this conveys the idea of a structure preserving map, for the idea of structure is not even present. Category theory shows that tbis definition is inadequate in other contexts, that is, there are maps which are monic and epic but not iso, e.g. in the category of posets or topological spaces. (I owe these last examples to an anonymous referee.) Moreover, the proper choice of maps in a given context is far from being dictated by set-theoretical considerations. For instance, in the category of topological spaces, one can take homeomorphisms or open maps or closed maps as the class of morphisms and this choice will have radical consequences on the categorical profiles of the resulting category.
[37]Bell has since published a book on topos theory in which he argues that topos theory should be considered to be fundamental in some sense. See Bell (1988).
[38]See Fourman (1980) for details.
[39]This is like Poincaré's argument against Russell according to which mathematical induction is synthetic, that is, it always will have to be *understood* independently of any of its logical reduction.
[40]Notice that the collection of natural numbers and its operations seem to be the most natural candidate here.
[41]See Barr and Wells (1985) for a sketch – no pun intended – of the geometric point of view and Bénabou (1967) and Street (1980) for higher-dimensional category theory.
[42]This should certainly be explored from a structuralist point of view. Unfortunately, none of the contemporary philosophers of mathematics who are trying to develop a structuralist philosophy of mathematics have considered that category theory might be the place to start. See, for instance, Resnik (1981, 1982, 1988), Parsons (1990) and Shapiro (1989).
[43]"I would speculate, though, that the physiological separation by the brain of the processing of spatial from the processing of the temporal thought supports the thesis that a complete unification of mathematics is not possible" (Mathias 1992, p. 11). This is a nice example of an argument where one concludes that the logical foundation of mathematics has to be such-and-such because of cognitive features. i.e. the relation of logical foundation

depends on the relation of cognitive foundation.

[44]This option might seem outrageous if only because there does not seem to be any historical precedent. However, a quick look at the history of the foundations of algebraic geometry in this century should convince anyone that this is not true: the search for appropriate foundations in algebraic geometry went through various stages, most of which were simply left behind, e.g. Weil's work, when the notion of scheme was finally developed by Grothendieck.

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