# NMAI057 – Linear algebra 1

## Tutorial 9 – with solutions

## Row space, column space, and kernel

Date: December 1, 2021 TA: Denys Bulavka

**Problem 1.** Compute the dimension and find the basis for the row space $\mathcal{R}(A)$, the column space $\mathcal{C}(A)$, and the kernel $\text{Ker}(A)$ of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

***Solution:***
First, we transform the matrix to the RREF:

$$\begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The non-zero rows in $\text{RREF}(A)$ form a basis of the row space $\mathcal{R}(A)$, i.e., the vectors $(1, 2, 0, 1)^T, (0, 0, 1, 1)^T$.

We choose the basis of the column space $\mathcal{C}(A)$ from the columns of the matrix $A$ by selecting the columns with the same index as the ones that have a pivot in $\text{RREF}(A)$, i.e., the first and third columns of $A$ form a basis of $\mathcal{C}(A)$.

We find the basis of the kernel of $A$ by solving the system $Ax = o$. Any solution of this system can be expressed parametrically in terms of $x_2, x_4 \in \mathbb{R}$ as

$$(-2x_2 - x_4, x_2, -x_4, x_4)^T = (-2, 1, 0, 0)^T x_2 + (-1, 0, -1, 1)^T x_4.$$

Thus, a basis of $\text{Ker}(A)$ is formed by the vectors $(-2, 1, 0, 0)^T, (-1, 0, -1, 1)^T$.

**Problem 2.** Over $\mathbb{R}$, $\mathbb{Z}_5$, and $\mathbb{Z}_7$, decide and justify whether for $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ it holds that

(a) $(1, 2)^T \in \text{Ker}(A)$,

(b) $(1, 2)^T \in \mathcal{C}(A)$.

***Solution:***
Recall the definition of kernel and the column space

$$\text{Ker}(A) = \{x \in \mathbb{F}^n : Ax = o\},$$
$$\mathcal{C}(A) = \text{span}\{A_{*1}, \ldots, A_{*n}\} = \{Ax : x \in \mathbb{F}^n\}.$$

Thus, it is sufficient to justify whether the vector $(1,2)^T$ is a solution for the system $Ax = o$ over the given field, respectively whether $Ax = (1,2)^T$ for some $x \in \mathbb{F}^2$.

Over $\mathbb{R}$:

(a) $(1,2)^T$ is not an element of $\mathrm{Ker}(A)$ since

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) $(1,2)^T$ is an element of $\mathcal{C}(A)$ since for

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{1}{5} \end{array}\right),$$

there exists a solution. Specifically, $(1,2)^T = \frac{3}{5}(1,3)^T + \frac{1}{5}(2,1)^T$.

Over $\mathbb{Z}_5$:

(a) $(1,2)^T$ is an element of $\mathrm{Ker}(A)$ since

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) $(1,2)^T$ is not an element of $\mathcal{C}(A)$ since for

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 4 \end{array}\right)$$

there is no solution over $\mathbb{Z}_5$.

Over $\mathbb{Z}_7$:

(a) $(1,2)^T$ is not an element of $\mathrm{Ker}(A)$ since

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) $(1,2)^T$ is an element of $\mathcal{C}(A)$ since for

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right),$$

there exists a solution over $\mathbb{Z}_7$. Specifically, $(1,2)^T = 2(1,3)^T + 3(2,1)^T$ over $\mathbb{Z}_7$.

**Problem 3.** Construct a matrix $A$ such that:

(a) $\mathcal{R}(A)$ contains vectors $(1,1)^T, (1,2)^T$ and $\mathcal{C}(A)$ contains $(1,0,0)^T$, $(0,0,1)^T$.

(b) The basis of both $\mathcal{R}(A)$ and $\mathcal{C}(A)$ is $(1,1,1)^T$ and the basis of $\mathrm{Ker}(A)$ is $(1,-2,1)^T$.

***Solution:***

(a) The dimensions of the vectors in the row space and the column space imply that the matrix is of order $3 \times 2$. The required properties are satisfied, fir example, by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) In this case, we should construct a $3 \times 3$ matrix for which

$$\dim \mathcal{R}(A) = \dim \mathcal{C}(A) = \operatorname{rank}(A) = \dim \operatorname{Ker}(A) = 1.$$

However, by the theorem about the relationship between rank and the dimension of the kernel of a matrix, for all matrices $A \in \mathbb{F}^{m \times n}$ it must hold that

$$\dim \operatorname{Ker}(A) + \operatorname{rank}(A) = n.$$

We conclude that a matrix satisfying the required properties does not exist.

**Problem 4.** Decide and justify whether for all $A, B \in \mathbb{R}^{n \times n}$ it holds that

(a) $\mathcal{C}(A) = \mathcal{C}(B)$ implies $\operatorname{RREF}(A) = \operatorname{RREF}(B)$,

(b) $\operatorname{RREF}(A) = \operatorname{RREF}(B)$ implies $\mathcal{C}(A) = \mathcal{C}(B)$.

***Solution:***

(a) The statement does not hold. For example, the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

have equal column spaces

$$\operatorname{span}\{(1,0)^T, (0,0)^T\} = \mathcal{C}(A) = \mathcal{C}(B) = \operatorname{span}\{(0,0)^T, (1,0)^T\},$$

but their RREFs are distinct (they are both already in RREF).

(b) Neither the reverse implication holds. For example, the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

have $\operatorname{RREF}(A) = \operatorname{RREF}(B) = A$ but

$$\operatorname{span}\left\{(1,0)^T, (0,0)^T\right\} = \mathcal{C}(A) \neq \mathcal{C}(B) = \operatorname{span}\left\{(0,1)^T, (0,0)^T\right\}.$$

**Problem 5.** Choose a basis $B$ of $V = \text{span}\{v_1, v_2, v_3, v_4\}$ from vectors

$$v_1 = (3,1,5,4)^T,\ v_2 = (2,2,3,3)^T,\ v_3 = (1,-1,2,1)^T,\ v_4 = (1,3,1,1)^T.$$

For the vectors not in your basis $B$, compute their coordinates w.r.t. $B$.

***Solution:***
We form a matrix using the given vectors as columns and reduced it to the RREF

$$\begin{pmatrix} 3 & 2 & 1 & 1 \\ 1 & 2 & -1 & 3 \\ 5 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The columns containing pivots are the first, third, and fourth one. Thus, the basis of $\mathcal{C}(A) = V$ is formed by the vectors $v_1 = (3,1,5,4)^T$, $v_2 = (2,2,3,3)^T$ a $v_4 = (1,3,1,1)^T$.

From the third column of $\text{RREF}(A)$ we get the coordinates of $v_3$ w.r.t. the basis $B = \{v_1, v_2, v_4\}$, it holds that

$$v_3 = (1,-1,2,1)^T = 1 \cdot (3,1,5,4)^T + (-1) \cdot (2,2,3,3)^T,$$

and $[v_3]_B = (1,-1,0)$.

**Problem 6.** Decide and justify whether for all $A, B \in \mathbb{R}^{m\times n}$ it holds that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.
*(Hint: What is the relationship between $\mathcal{C}(A) + \mathcal{C}(B)$ and $\mathcal{C}(A+B)$?)*

***Solution:***
Consider the space generated by the union of the columns of matrices $A$ and $B$, i.e., the space $\mathcal{C}(A) + \mathcal{C}(B)$. The dimension of this space is at most

$$\dim \mathcal{C}(A) + \dim \mathcal{C}(B) = \text{rank}(A) + \text{rank}(B).$$

Moreover, the space $\mathcal{C}(A) + \mathcal{C}(B)$ contains all the vectors generated by the columns of matrix $A+B$. Thus, $\mathcal{C}(A+B)$ is a subspace of $\mathcal{C}(A)+\mathcal{C}(B)$. Therefore, we can conclude that

$$\text{rank}(A+B) = \dim \mathcal{C}(A+B) \leq \dim \mathcal{C}(A) + \mathcal{C}(B) \leq \text{rank}(A) + \text{rank}(B).$$

**Problem 7.** In terms of inclusion, what is the relationship between $\text{Ker}(AB)$ and $\text{Ker}(B)$ for matrices

(a) $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$,

(b) $A \in \mathbb{R}^{n\times n}$ regular and $B \in \mathbb{R}^{n\times p}$?

***Solution:***

(a) For all $x \in \mathrm{Ker}(B)$, by definition of kernel, it holds that $Bx = o$. Moreover, $x$ is contained in the kernel of $AB$ since

$$(AB)x = A(Bx) = Ao = o,$$

and we get the inclusion $\mathrm{Ker}(B) \subseteq \mathrm{Ker}(AB)$. The reverse inclusion does not hold in general. For example, for $A = 0_{n\times n}$ and $B = I_n$, the vector $y = (1, 0, \ldots, 0)^T$ is contained in the kernel of $AB$ but not in the kernel of $B$.

(b) For regular matrices $A$, the reverse inclusion $\mathrm{Ker}(AB) \subseteq \mathrm{Ker}(B)$ holds and, thus, $\mathrm{Ker}(AB) = \mathrm{Ker}(B)$. For all $x \in \mathrm{Ker}(AB)$, it holds that $(AB)x = o$. By regularity of $A$, there exists and inverse matrix $A^{-1}$ such that

$$Bx = (A^{-1}A)Bx = A^{-1}((AB)x) = A^{-1}o = o,$$

which implies that $x \in \mathrm{Ker}(B)$.