NMAI057 – Linear algebra 1

Tutorial 7

Fields

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Problem 1. Simplify the following expressions:

(a) $((2^{-1}+1)4)^{-1}, 4/3$ over $\mathbb{Z}_5$,

(b) $6+7, -7, 6\cdot 7, 7^{-1}, 6/7$ over $\mathbb{Z}_{11}$.

Problem 2. Over $\mathbb{Z}_5$, find the set of all solutions of the system

$$\begin{aligned} 3x+2y+z&=1\\ 4x+y+3z&=3 \end{aligned}$$

and compute its cardinality.

Problem 3. Find the multiplicative inverses 9^{-1} and 12^{-1} in $\mathbb{Z}_{31}$.

Problem 4. Over $\mathbb{Z}_7$, compute the matrix power A^{100} for $A = \left(\begin{smallmatrix} 3 & 2\\ 1 & 4\end{smallmatrix}\right)$.

Problem 5. For $n \in \mathbb{N}$ and an asociative operation $\cdot$ let $a^n = a\cdot a\cdot \ldots \cdot a$, where the element a appears n times in the product.

- Determine values $2^{101}, 3^{1\,001}$ and $4^{1\,000\,001}$ in the field $\mathbb{Z}_{17}$.
- Determine $5^{100}, 8^{200}, 11^{300}$ and 18^{400} in the field $\mathbb{Z}_{19}$.

Problem 6. Solve the following system of equations over $\mathbb{Z}_5, \mathbb{Z}_7$ and $\mathbb{R}$.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 4x_3 & = & 3\\ 3x_1 & + & x_2 & + & 2x_3 & = & 4\\ 2x_1 & + & 4x_2 & + & x_3 & = & 3 \end{array}$$

Problem 7. Invert the following matrices over fields $\mathbb{Z}_3$ and $\mathbb{Z}_5$

- $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1\\ 2 & 0 & 1 & 1\\ 2 & 1 & 0 & 0\\ 1 & 2 & 1 & 0 \end{pmatrix}$.
- $\mathbf{B} = \begin{pmatrix} 0 & 2 & 2 & 1\\ 1 & 0 & 2 & 0\\ 2 & 1 & 0 & 2\\ 2 & 2 & 1 & 1 \end{pmatrix}$.

- $\mathbf{C} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.
- $\mathbf{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$.
- $\mathbf{E} = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 \end{pmatrix}$.

Problem 8. Invert the following matrix over $\mathbb{Z}_{11}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Problem 9. Find a matrix $\mathbf{A}$, that over $\mathbb{Z}_5$ satisfies

$$\mathbf{A} \begin{pmatrix} 4 & 4 & 0 & 1 \\ 3 & 1 & 2 & 2 \\ 2 & 3 & 1 & 3 \\ 3 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 2 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$