- **Groups:** $(G, \circ)$ $\quad \circ : G \times G \longrightarrow G$, $\circ(a,b) = a \circ b$.

  a) associative: $\forall a,b,c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$.

  b) idensity elem. for $\circ$: $\exists e \in G$ st $\forall a \in G$: $e \circ a = a \circ e = a$.

  c) inverse for $\circ$: $\forall a \in G$ $\exists b \in G$ st $a \circ b = b \circ a = e$.

- $(\mathbb{Z}, +)$ $\quad \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
  - a) $a, b, c \in \mathbb{Z}$, $\quad a + (b+c) = (a+b) + c$.
  - b) $e \in \mathbb{Z}$, $\forall a \in \mathbb{Z}$ $\quad a + e = a \Rightarrow \boxed{e = 0}$

    $e + a = 0 + a = a$.
  - c) $a \in G$, $b \in G$, $\quad a + b = e = 0 \Rightarrow \boxed{b = -a}$

    $b + a = -a + a = 0 = e$. ✓

- $(\mathbb{N}, +)$; $\mathbb{N} = \{1, 2, 3, \ldots\}$.
  - a) $a + (b + c) = (a + b) + c$
  - b) $e \in \mathbb{N} \nexists \quad a + e = a \Rightarrow \boxed{e = 0}$ ⇝ it is not a group.

- $(\mathbb{N} \cup \{0\}, +)$
  - c) $a \in \mathbb{N}, \; b \in \mathbb{N} \nexists \quad a + b = 0 \Rightarrow b = -a \notin \mathbb{N} \cup \{0\}$.

    it is not a group either.

Permutations

- a permutation $p : \{1, .., n\} \to \{1, .., n\}$ bijection.
- $(\{p : \{1, .., n\} \to \{1, .., n\}\}, \circ) = S_n$

  ↑ composition of functions

  [neutral element: $i : \{1, .., n\} \to \{1, .., n\}$, $i(j) = j$]

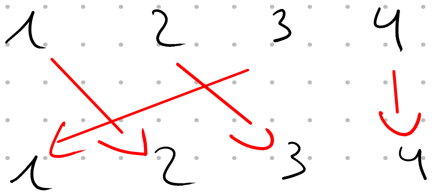
- $p, \varphi \in S_n \Rightarrow (p \circ \varphi)(i) = p(\varphi(i))$
- How do we describe a permutation?

table

i	1	2	3	4
$p(i)$	2	3	1	4

list : $(1\ 3\ 2)(4)$

bipartite graph

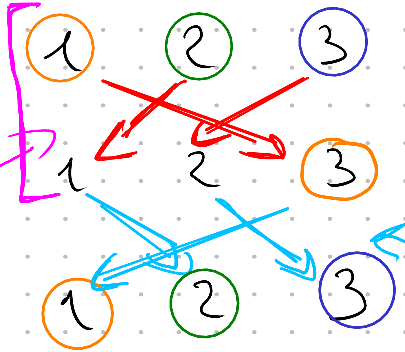


permutation matrix

$$P_{ij} = \begin{cases} 1 & \text{if } p(i) = j \\ 0 & \text{otherwise} \end{cases}$$

$(1\ 2\ 3)$

$(2\ 1\ 3)$



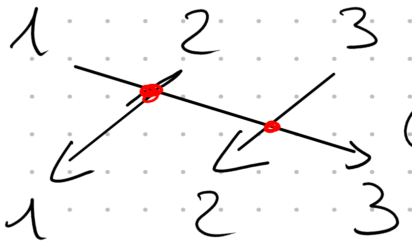
φ and p:

$[(2\ 1\ 3) \circ (1\ 2\ 3) = i]$

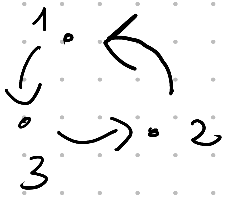
$(1\ 2\ 3) \circ (1\ 2\ 3) = (1\ 2\ 3)^2$

$(1\ 2\ 3)$

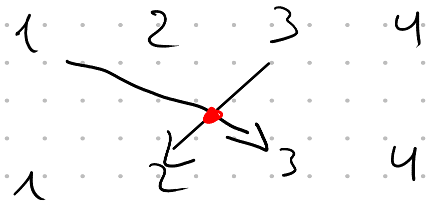
$$(\varphi \circ p)(1) = \varphi(p(1)) = \varphi(3) = 1.$$

- 

$(-1)^2 = 1$

- 

- <u>inversion of a permutation p</u>: (i, j) st $i < j$ & $p(i) > p(j)$



$\text{sgn}(p) = (-1)^{\#\text{ inversions of } p}$.

$\text{sgn}(p) = \text{sgn}(p^{-1})$

$= (-1)^{\#\text{ even cycles of } p}$.