On the significance of the principle of excluded middle in mathematics, especially in function theory

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The text below is the translation of an address delivered in German on 21 September 1923 at the annual convention of the Deutsche Mathematiker-Vereinigung in Marburg an der Lahn. It had been delivered in Dutch at the 22nd Vlaamsch Natuur- en Geneeskundig Congres, in Antwerp in August 1923, in an approximately similar form (*Brouwer 1923a*).

§ 1 shows how the principles of logic, which have their origin in finite mathematics, came to be applied to discourse about the physical world and then to nonfinite mathematics; but in that last field there is not necessarily a justification for each of these principles. In particular, such a justification seems to be lacking for the principle of excluded middle and that of double negation.

§ 2 shows how several important results of classical analysis become unjustified once the principle of excluded middle is abandoned. Here Brouwer's critique is essentially negative, being based on counterexamples to classical theorems; but elsewhere he investigates which fragments of the Bolzano-Weierstrass theorem can be preserved in intuitionistic analysis (*1919*, sec. 1, and *1952a*; see also *Heyting 1956*, arts. 3.4.4 and 8.1.3) and gives an intuitionistic form of the Heine-Borel theorem (*1926a* and *1926b*; see also *Heyting 1956*, art. 5.2.2). There are further counterexamples to theorems of classical analysis in *Brouwer 1928a*.

§ 3 is an example of the "splitting" of a classical notion, that of a convergent sequence, into several overlapping but distinct intuitionistic notions, here positively convergent sequence, negatively convergent sequence, and nonoscillating sequence. These notions were further investigated by one of Brouwer's disciples, M. J. Belinfante, and we refer the reader to Belinfante's papers listed below, p. 630. In order to avoid a number of complications that arise in the theory of infinite sequences as elaborated by Brouwer and Belinfante, J. G. Dijkman found it convenient (*1948*) to introduce the notions of strictly negatively convergent sequence and of strictly nonoscillating sequence.

Two notes, "Addenda and corrigenda" and "Further addenda and corrigenda", published by Brouwer in 1954, are appended to the 1923 paper. They reflect the development of Brouwer's ideas in the intervening years. In the main paper below (*1923b*) Brouwer had introduced an infinite sequence whose definition depends upon the occurrence of a certain finite sequence of digits in the decimal expansion of π. In *1948* he introduced an infinitely proceeding sequence whose definition depends upon whether a certain mathematical problem has, or has not, been solved at a certain time: let α be a mathematical assertion that so far has not been tested, that is, such that neither $\neg\alpha$ nor $\neg\neg\alpha$ has been proved; then, if between the choice for c_{n-1} and the choice for c_n "the creating subject has experienced either the truth or the absurdity of α" (*1948*, p. 1246), a certain value is chosen for c_n; otherwise, another value is chosen for c_n. This method of definition, by which the choices for the constituents of an infinitely proceeding sequence "may, at any stage, be made to depend on possible future mathematical experiences of the creating subject" (*1953*, p. 2), allowed Brouwer to offer new counterexamples to classical theorems, in particular in analysis (*1948a*, *1948b*, *1949*, *1949a*, *1950*, *1950a*, *1951*, and *1952a*). It is in these conditions that he came to write the two appendices, *1954* and *1954a*; *1954b* and *1954c* constitute a sequel to *1954a*.

The translation of the main paper (*1923b*) is by Stefan Bauer-Mengelberg and the editor, and it is printed here with the kind permission of Professor Brouwer and Walter de Gruyter and Co. The first appended paper (*1954*) was translated by Stefan Bauer-Mengelberg, Claske M. Berndes Franck, Dirk van Dalen, and the editor; the second appended paper (*1954a*) was translated by Stefan Bauer-Mengelberg, Dirk van Dalen, and the editor.

§ 1

Within a specific finite "main system" we can always *test* (that is, either prove or reduce to absurdity) properties of systems, that is, test whether systems can be mapped, with prescribed correspondences between elements, into other systems; for the mapping determined by the property in question can in any case be performed in only a finite number of ways, and each of these can be undertaken by itself and pursued either to its conclusion or to a point of inhibition. (Here the principle of mathematical induction often furnishes the means of carrying out such tests without individual consideration of every element involved in the mapping or of every possible way in which the mapping can be performed; consequently the test even for systems with a very large number of elements can at times be performed relatively rapidly.)

On the basis of the testability just mentioned, there hold, for properties conceived within a specific finite main system, the *principle of excluded middle*, that is, the principle that for every system every property is either correct [[richtig]] or impossible, and in particular the *principle of the reciprocity of the complementary species*, that is, the principle that for every system the correctness of a property follows from the impossibility of the impossibility of this property.

If, for example, the union $\mathfrak{S}(p, q)$ of two mathematical species[1] p and q contains at least eleven elements, it follows on the basis of the principle of excluded middle (which in this case appears as "principle of disjunction") that either p or q contains at least six elements.

Likewise, if we have proved in elementary arithmetic that, whenever none of the positive integers $a_1, a_2, \ldots, a_n$ is divisible by the prime number c, the product $a_1 a_2 a_3 \ldots a_n$ is not divisible by c either, it follows on the basis of the principle of the reciprocity of the complementary species that, if the product $a_1 a_2 a_3 \ldots a_n$ is divisible by the prime number c, at least one of the factors of the product is divisible by c.

[1] [[For the definition of "species" see below, p. 454.]]

For properties derived within a specific finite main system by means of the principle of excluded middle it is always certain that we can arrive at their empirical corroboration if we have a sufficient amount of time at our disposal.

It is a natural phenomenon, now, that numerous objects and mechanisms of the world of perception, considered in relation to extended complexes of facts and events, can be mastered if we think of them as (*possibly partly unknown*) finite discrete systems that for specific known parts are bound by specific laws of temporal concatenation. Hence the laws of theoretical logic, including the principle of excluded middle, are applicable to these objects and mechanisms in relation to the respective complexes of facts and events, even though here a complete empirical corroboration of the inferences drawn is usually materially excluded a priori and there cannot be any question of even a partial corroboration in the case of (juridical and other) inferences about the past. To this incomplete verifiability of inferences that are nevertheless considered irrefutably correct, as well as to our partial ignorance of the representing finite systems and to the fact that theoretical logic is applied more often and by more people to such material objects than to mathematical ones we must probably attribute the fact that an a priori character has been ascribed to the laws of theoretical logic, including the principle of excluded middle, and that one lost sight of the conditions of their applicability, which lie in the projection of a finite discrete system upon the objects in question, so that one even went so far as to look to the laws of logic for a deeper justification of the completely primary and autonomous mental activity ⟦Denkhandlung⟧ that the mathematics of finite systems represents. Accordingly, in the logical treatment of the world of perception the appearance of a contradiction never led us to doubt that the laws of logic were unshakable but only to modify and complete the mathematical fragments projected upon this world.

An a priori character was so consistently ascribed to the laws of theoretical logic that until recently these laws, including the principle of excluded middle, were applied without reservation even in the mathematics of infinite systems and we did not allow ourselves to be disturbed by the consideration that the results obtained in this way are in general not open, either practically or theoretically, to any empirical corroboration. On this basis extensive incorrect theories were constructed, especially in the last half-century. The contradictions that, as a result, one repeatedly encountered gave rise to the *formalistic critique*, a critique which in essence comes to this: the *language accompanying the mathematical mental activity* is subjected to a mathematical examination. To such an examination the laws of theoretical logic present themselves as operators acting on primitive formulas or axioms, and one sets himself the goal of transforming these axioms in such a way that the linguistic effect of the operators mentioned (which are themselves retained unchanged) can no longer be disturbed by the appearance of the linguistic figure of a contradiction. We need by no means despair of reaching this goal,[2] but nothing of mathematical value will thus be gained: an incorrect theory, even if it cannot be inhibited by any contradiction that would refute it, is none the less incorrect, just as a criminal policy is none the less criminal even if it cannot be inhibited by any court that would curb it.

[2] For the unjustified application of the principle of excluded middle to properties of well-constructed mathematical systems can never lead to a contradiction (see *Brouwer 1908*, ⟦p. 157, or *1919a*, p. 11⟧).

§ 2

The following two fundamental properties, which follow from the principle of excluded middle, have been of basic significance for this incorrect "logical" mathematics of infinity ("logical" because it makes use of the principle of excluded middle), especially for the *theory of real functions* (developed mainly by the Paris school):

1. *The points of the continuum form an ordered point species*;[3]
2. *Every mathematical species is either finite or infinite.*[4]

The following example shows that the first fundamental property is incorrect. Let d_ν be the νth digit to the right of the decimal point in the decimal expansion of π, and let $m = k_n$ if, as the decimal expansion of π is progressively written, it happens at d_m for the nth time that the segment $d_m d_{m+1} \ldots d_{m+9}$ of this decimal expansion forms the sequence 0123456789. Further, let $c_\nu = (-\frac{1}{2})^{k_1}$ if $\nu \geqq k_1$, otherwise let $c_\nu = (-\frac{1}{2})^\nu$; then the infinite sequence $c_1, c_2, c_3, \ldots$ defines a real number r for which none of the conditions $r = 0$, $r > 0$, or $r < 0$ holds.[5]

When the first fundamental property ceases to hold, the Paris school's notion of integral, the notion of L-integral, as it is called, ceases to be useful, because this notion of integral is bound to the notion "measurable function" and, according to the above, not even a constant function satisfies the conditions of "measurability". For in the case of the function $f(x) = r$, where r represents the real number defined above, the values of x for which $f(x) > 0$ do not form a measurable point species.[6]

That the second fundamental property is incorrect is seen from the example provided by the species of the positive integers k_n defined above.

When the second fundamental property ceases to hold, so does the "extended disjunction principle", according to which, if a fundamental sequence of elements is contained in the union $\mathfrak{S}(p, q)$ of two mathematical species p and q, either p or q contains a fundamental sequence of elements; and when the extended disjunction principle ceases to hold, so does the Bolzano-Weierstrass theorem, which rests upon it and according to which every bounded infinite point species has a limit point.

The following two theorems are less basic and simple than the fundamental properties mentioned, yet they are equally indispensable for the construction of the "logical" theory of functions.

1. *Every continuous function $f(x)$ defined everywhere in a closed interval i possesses a maximum, that is, an abscissa value x_1 having a neighborhood α such that $f(x_1) \geqq f(x)$ for every x that belongs to the intersection of α and i.*

[3] That is, if on the one hand $a < b$ either holds or is impossible, or on the other $a > b$ either holds or is impossible, then one of the conditions $a < b$ or $a > b$ or $a = b$ holds.

[4] For according to the principle of excluded middle a species s either is finite or cannot possibly be finite. In the latter case s possesses an element, e_1; for otherwise, on the basis of the principle of excluded middle, s could not possibly possess an element and would therefore be finite, which is excluded. Furthermore s possesses an element, e_2, distinct from e_1; for otherwise s would not possibly possess an element distinct from e_1 and would therefore be finite, which is excluded. Continuing in this manner, we show that s possesses a fundamental sequence of distinct elements, $e_1, e_2, \ldots$. [[For the definition of "fundamental sequence" see below, p. 455.]]

[5] Of course, we can also define r by means of any other property x whose existence or impossibility can be derived for every definite positive integer, while we can neither determine a positive integer that possesses x nor prove the impossibility of x for all positive integers.

[6] However, the notion of R-integral, that is, the notion of Riemann integral, can be applied to $f(x)$ without further ado.

The incorrectness of this theorem appears from the following example: If we enumerate the irreducible binary fractions between 0 and 1 (excluding 0 and 1) by means of a fundamental sequence $\delta_1, \delta_2, \ldots$ in the ordinary way, that is, so that any fraction follows all those with a smaller denominator and fractions with the same denominator are ordered according to the magnitude of the numerator, if we assign to k_1 the same meaning as above, if by $f_n(x)$ we understand the function that has the value 2^{-n} for $x = \delta_n$ and vanishes for $x = 0$ as well as for $x = 1$, while it remains linear between $x = 0$ and $x = \delta_n$ as well as between $x = \delta_n$ and $x = 1$, and if we put $g_n(x) = f_n(x)$ for $n = k_1$, otherwise $g_n(x) = 0$, then the continuous function

$$g(x) = \sum_{n=1}^{\infty} g_n(x),$$

which is defined everywhere in the closed unit interval, possesses no maximum.

2. (Heine-Borel covering theorem.) *If a neighborhood is assigned to every point core*[7] *of the point species A formed by the points and the limit points of a bounded entire*[8] *point species B, then the whole point species A can be covered by a finite number of these neighborhoods.*

The incorrectness of this theorem appears from the following example: If we choose for B the number sequence $c_1, c_2, c_3, \ldots$, defined above, while we assign to the number c_ν, for $\nu \geqq k_1$, the interval $(c_\nu - 2^{-k_1-2}, c_\nu + 2^{-k_1-2})$, otherwise the interval $(c_\nu - 2^{-\nu-2}, c_\nu + 2^{-\nu-2})$, and to a limit point e (if any) of the sequence the interval $(e - \frac{1}{2}, e + \frac{1}{2})$, then A cannot be covered by a finite number of these neighborhoods.[9]

In view of the fact that the foundations of the logical theory of functions are indefensible according to what was said above, we need not be surprised that a large part of its results becomes untenable in the light of a more precise critique. As an example, we shall refute one of the best-known classical theorems in this domain, namely, the theorem that a monotonic continuous function defined everywhere is "almost everywhere" differentiable, by constructing a monotonic continuous function that is defined everywhere in the closed unit interval but is nowhere differentiable.

Let $0 \leqq x_1 < x_2 \leqq 1$. By the *elementary function corresponding to the interval* (x_1, x_2) we shall understand the continuous function, defined everywhere in the closed unit interval, that, for $x_1 \leqq x \leqq x_2$, is equal to

$$\frac{x_2 - x_1}{2\pi} \sin 2\pi \frac{x - x_1}{x_2 - x_1}$$

and, for $0 \leqq x \leqq x_1$ and $x_2 \leqq x \leqq 1$, is equal to 0; by $\lambda', \lambda'', \lambda''', \ldots$ we shall understand the intervals $(a/2^n, (a + 2)/2^n)$ (where a and n denote positive integers) belonging to the closed unit interval and enumerated in the customary way; and by $f_n(x)$ we shall understand the elementary function corresponding to $\lambda^{(n)}$. Furthermore we

7 [[For the definition of "point core" see below, p. 458.]]

8 [["The species of the points that coincide with points of the point species Q is called the *completing* [[*ergänzende*]] *point species* or, for short, the *completion* [[*Ergänzung*]] of Q. A point species that is identical with its completion is called an *entire* [[*ganze*]] Punktspecies." (*Brouwer 1919*, p. 6.) For the definition of "coincide" see below, p. 458, and for that of "identical with", p. 454.]]

9 Nor does the theorem hold for a closed bounded entire point species A. Counterexample: take for A a species of abscissas $(-2)^{-\nu}$ such that an abscissa $(-2)^{-\nu}$ belongs to A if and only if a natural number k_1 satisfying the characterization above is known and ν is a natural number $\leqq k_1$; then with each abscissa that may belong to A associate the same interval as in the text.

assign to k_1 the same meaning as above; we put $g_1(x) = x$ and (for $n \geqq 2$) $g_n(x) = f_n(x)$ for $n = k_1$, otherwise $g_n(x) = 0$. Then the function

$$g = \sum_{n=1}^{\infty} g_n$$

is a monotonic continuous function that is defined everywhere in the closed unit interval but is nowhere differentiable.

§ 3

As an example illustrating the fact that even older and more firmly consolidated theories in the field of the mathematics of infinity are affected by the rejection of the principle of excluded middle and the consequent rejection of the Bolzano-Weierstrass theorem, even if in much smaller measure than the theory of real functions, we take the notion of convergence of infinite series.

Let us say that an infinite series $u_1 + u_2 + u_3 + \cdots$ with real terms, for which the sum of the first n terms is denoted by s_n, is *nonoscillating* if for every $\varepsilon > 0$ it has been established that it is impossible to have at the same time an infinite sequence of positive integers $n_1, n_2, n_3, \ldots$ increasing beyond all bounds and an infinite sequence of positive integers $m_1, m_2, m_3, \ldots$ such that

$$|s_{n_\nu + m_\nu} - s_{n_\nu}| > \varepsilon \quad \text{for every } \nu;$$

then according to the classical theory on the basis of the principle of excluded middle such a nonoscillating series is:

1. *Negatively convergent*, that is, there exists a real number s with the property that for every $\varepsilon > 0$ it has been established that it is impossible to have an infinite sequence of positive integers $n_1, n_2, n_3, \ldots$ increasing beyond all bounds such that

$$|s - s_{n_\nu}| > \varepsilon \quad \text{for every } \nu;$$

2. *Bounded*, that is, there exist two real numbers g_1 and g_2 such that

$$g_1 < s_n < g_2 \quad \text{for every } n;$$

3. *Positively convergent*, that is, there exists a real number s with the property that for every $\varepsilon > 0$ there exists a positive integer n_ε such that

$$|s - s_n| < \varepsilon \quad \text{for every } n > n_\varepsilon.$$

Let us now consider the following five nonoscillating series (where k_1 again has the same meaning as above):

(a) $u_n = 1/2^n$ for every n;
(b) $u_n = 2 + 1/2^n$ for $n = k_1$, $u_n = -2 + 1/2^n$ for $n = k_1 + 1$, otherwise $u_n = 1/2^n$;
(c) $u_n = n + 1/2^n$ for $n = k_1$, $u_n = -n + 1/2^n$ for $n = k_1 + 1$, otherwise $u_n = 1/2^n$;
(d) $u_n = 1$ for $n = k_1$, otherwise $u_n = 1/2^n$;
(e) $u_n = n$ for $n = k_1$, otherwise $u_n = 1/2^n$.

The series (a) turns out to be positively convergent and therefore also negatively convergent and bounded; the series (b) to be negatively convergent and bounded, but not positively convergent; the series (c) to be negatively convergent, but not bounded and therefore not positively convergent either; the series (d) to be bounded, but not negatively convergent and therefore not positively convergent either; the series (e), finally, to be not bounded, not negatively convergent, and not positively convergent.

To illustrate the consequences of the distinction made above we shall consider the Kummer convergence criterion, which reads as follows: "If $B_1, B_2, \ldots$ are positive numbers and if, for the infinite series of positive terms $r = u_1 + u_2 + u_3 + \cdots$, we have

$$\lim \left\{ B_n \frac{u_n}{u_n + 1} - B_{n+1} \right\} > 0,$$

then r is positively convergent".

The proof of this convergence criterion is customarily carried out as follows.

On the basis of what has been assumed we select M and k in such a way that, for $n \geqq M$,

$$B_n \frac{u_n}{u_n + 1} - B_{n+1} > k,$$

$$B_n u_n - B_{n+1} u_{n+1} > k u_{n+1},$$

$$B_n u_n - B_{n+p} u_{n+p} > k(u_{n+1} + \cdots + u_{n+p}),$$

$$u_{n+1} + \cdots + u_{n+p} < \frac{B_n u_n}{k},$$

whence *boundedness* follows for the series $r_n = u_{n+1} + u_{n+2} + \cdots$ $(n \geqq M)$ and therefore also for the series $r = u_1 + u_2 + \cdots$. On the basis of this boundedness the series r is then declared to be not only nonoscillating, which is permitted for a series of positive terms, but also positively convergent.

The last inference, however, rests upon the Bolzano-Weierstrass theorem and must be rejected along with it.

Pringsheim (*1916*, p. 378) offers an altogether different and more instructive proof. After he has proved the positive convergence of r for the case of the positive convergence as well as for the case of the positive divergence of $b = 1/B_1 + 1/B_2 + \cdots$, he assumes that the series b must be either positively convergent or positively divergent, and for this reason he declares that the general criterion has been proved.

But the assumption mentioned is inadmissible; for it, too, rests upon the Bolzano-Weierstrass theorem.

It is worth noting, now, that Kummer himself expressed (*1835*) his criterion only with the auxiliary condition $\lim B_n u_n = 0$ and that with this auxiliary condition the positive convergence of the series r is actually ensured by the criterion, as is immediately evident from the proof above.

That not only the derivations of the Kummer convergence criterion without any auxiliary condition are inadequate[10] but also the criterion itself is incorrect is shown

[10] The inadequacy of these derivations, in contradistinction to the correctness of the proof originally carried out by Kummer himself for the restricted criterion, was indicated to me by my student M. J. Belinfante as an example of the significance of the principle of excluded middle for the theory of infinite series.

by the series (d) above, which is neither positively convergent nor negatively convergent. For, if we determine the successive B_n for this series from the relations

$$B_1 = 4 \quad \text{and} \quad B_n \frac{u_n}{u_n + 1} - B_{n+1} = 1 \quad \text{for every } n,$$

all B_n turn out to be positive, so that the extended convergence criterion is satisfied here, although positive convergence does not exist. This *omission of the Kummer auxiliary condition*, which took place after Kummer and was prompted by Dini, has thus considerably curtailed the scope of the convergence criterion in question.

ADDENDA AND CORRIGENDA

(1954)

Regarding my paper "Over de rol van het principium tertii exclusi in de wiskunde, in het bijzonder in de functietheorie" (*1923a*), published thirty years ago in volume 2 of *Wis- en Natuurkundig Tijdschrift*, which has since been discontinued, I would now like to make the following remarks.

I. Page 1, line 4 ⟦above, page 335, line 1⟧, the term "to test" ⟦"toetsen" (*1923a*), or "prüfen" (*1923b*)⟧ is used for either proving or reducing to absurdity. In subsequent intuitionistic literature, however, a property of a mathematical entity is said to be "tested" if either its contradictoriness or its noncontradictoriness is ascertained, and "judged" ⟦"geoordeeld"⟧ if either its presence or its absurdity is ascertained.

II. Page 3, footnote (*) ⟦above, page 336, footnote 2⟧, the noncontradictoriness of applications of the principle of excluded middle to the attribution of a property E to a well-constructed mathematical system was pointed out. In subsequent intuitionistic literature, however, it became apparent that for the simultaneous application of the principle mentioned to the attribution of a property E to each element of a mathematical species S noncontradictoriness remains ensured only for finite S. For infinite S the simultaneous attribution mentioned can very well be contradictory.

III. Page 3, footnote (****) ⟦above, page 337, footnote 5⟧, for the construction, given in the text, of a real number r for which none of the relations $r = 0$, $r > 0$, and $r < 0$ holds, we allowed every property x for which neither a finite number possessing x nor the impossibility of x for every finite number is known. To this we must add the condition that x can be judged for every finite number.

IV. Page 4, line 18 up ⟦above, page 338, line 12⟧, the classical Heine-Borel covering theorem was formulated for an arbitrary "closed" bounded point species. The intuitionistic critique of this theorem that follows there should have been preceded by an exposition of the intuitionistic splitting of the classical notion "closed". For, if in a Cartesian or in a "located" ⟦"afgebakende"⟧ compact topological space R we understand by a *core* the species of the points that coincide with a given point, by an *accumulation core* of a core species Q a core of which every neighborhood contains an infinitely proceeding sequence of cores of Q that are mutually apart, and by a *limit core* of a core species Q a core of which every neighborhood contains a core of Q, if we then say that a core species Q containing all of its accumulation cores is *α-closed* and that a core species Q that contains all of its limit cores is *β-closed*, if, accordingly, we call the union of a core species Q and its accumulation cores the *α-closure of* Q and the species of limit cores of Q the *β-closure of Q*, if we take the formulation cited above of the classical Heine-Borel covering theorem as applying to "closed" bounded core species Q, then this formulation is intuitionistically correct only if by "closed" is meant "β-closed" and if, moreover, Q is a core species *located in R*, that is to say, it is from every core of R at a distance that is computable with unlimited accuracy. In particular, therefore, with regard to the number sequence $c_1, c_2, c_3, \ldots$ referred to on page 4, line 13 up [[above, page 338, line 17]], which is bounded and is located in the number continuum, the classical covering theorem is intuitionistically valid only for its *β-closure*, that is to say, for its union with its limit number, but not for its *α-closure*, referred to on page 4, line 13 up [[above, page 338, line 19]], that is to say, for its union with the number 0, *if this number should turn out to be identical with the limit number*. Nor is the classical covering theorem intuitionistically valid for number core species that are β-closed and bounded *but not located in the number continuum*, as, for example, the union of the number cores $p_1, p_2, p_3, \ldots$, in which $p_\nu = 1$ for $\nu < k_1$ and $p_\nu = -1$ for $\nu \geqq k_1$.

V. The example given on page 5, lines 1–13 [[above, page 338, line 8u, to page 339, line 5]], of a monotonic, continuous, nowhere differentiable function defined everywhere in the closed unit interval possesses these properties exclusively as a function of the (classical) continuum of approximations made according to a law, not as a function of the (intuitionistic) continuum of more or less freely proceeding approximations. A connection between monotonicity and differentiability of full functions of the intuitionistic continuum can be found in my *1923*, p. 24.

FURTHER ADDENDA AND CORRIGENDA

(1954a)

With reference to point V of my *1954*, pp. 104–105 [[above, pp. 341–342]], I give below an example of a *continuous, monotonic, nowhere differentiable, real, full function of the intuitionistic closed unit continuum K.*[1]

For a natural number n we understand by $\chi_n(x)$ the real function of K that for the "even n-cores"[2] $x = a/n$ (a being an integer and $0 \leqq a \leqq n$) is equal to 0, for the "odd n-cores" $x = (2a + 1)/2n$ (a being an integer and $0 \leqq a \leqq n$) is equal to $1/4n$, and for every a ($0 \leqq a \leqq n$) is linear between $x = a/n$ and $x = (2a + 1)/2n$ as well as between $x = (2a + 1)/2n$ and $x = (a + 1)/n$.[3] Further we put $\psi_1(x) \equiv x$ and, for $n \geqq 2$, f being an opaque fleeing property and $\kappa_1(f)$ being its critical number,[4] we

[1] [[For the definitions of "continuous", "full", and "unit continuum" see below, pp. 458–459; see also *Brouwer 1953*, p. 3, line 2u, to p. 4, line 6.]]

[2] [[For the definition of "core" see below, p. 458; see also *Brouwer 1953*, p. 3, line 2u, to p. 4, line 6.]]

[3] [[From the intuitionistic point of view the definition of $\chi_n(x)$ does not seem unobjectionable; see Remark in 2.2.8 of *Heyting 1956*, p. 27.]]

[4] [["We shall call a hypothetical property f of natural numbers a *fleeing property* if it satisfies the following conditions:

(1) For each natural number it can be decided either that it possesses the property f or that it cannot possibly possess the property f;

(2) No method is known for calculating a natural number possessing the property f;

(3) The assumption of existence of a natural number possessing the property f is not known to lead to an absurdity.

In particular, a fleeing property is called *opaque* if the assumption of existence of a natural put $\psi_n(x) \equiv \chi_n(x)$ if $n = \kappa_1(f)$, otherwise $\psi_n(x) \equiv 0$. Then

$$\psi(x) \equiv \sum_{\nu=1}^{\infty} \psi_\nu(x)$$

is a continuous, monotonic, nowhere differentiable, real, full function of K.

For one must take into account the possibility (α) that at some time it turns out that $\kappa_1(f)$ is nonexistent, so that, for all values of x, $\psi(x)$ possesses an ordinary derivative equal to 1.

But one must also take into account the possibility (β) that at some time a natural number $m = \kappa_1(f)$ will be found. In that case $\psi(x)$ has, for all values of x that lie apart[5] from the m-cores, an ordinary derivative, either equal to 3/2 or equal to 1/2; for all even m-cores x it has a right derivative (nonexistent for $x = 1$) equal to 3/2, and a left derivative (nonexistent for $x = 0$) equal to 1/2; and for all odd m-cores x it has a right derivative equal to 1/2 and a left derivative equal to 3/2, while for every value of x the possibility must be taken into account that at some time it shall turn out either to be an m-core or to lie apart from the m-cores.

Therefore, with respect to the existence of an ordinary derivative, or of a right and a left derivative, of $\psi(x)$ one must, *for every value of x, take into account possibilities lying mutually apart, so that for no single value of x an ordinary derivative can be calculated.*

By the nature of the case this function $\psi(x)$ is not "completely differentiable" in the sense of *Brouwer 1923*, § 3, p. 20.[6]

So far as the function $g(x)$, mentioned in *Brouwer 1923a*, p. 5 [[above, p. 339]], is concerned, it must, according to the explanations that follow below, be abandoned as an example of a continuous, monotonic, nowhere differentiable function, *even for the classical* closed unit continuum K_r.[7]

§ 2

By a $k^{(\nu)}$ we understand a closed $\lambda^{(4\nu+1)}$-interval;[8] for $\nu \geqq 0$, by an $h^{(\nu)}$ we understand a $k^{(\nu)}$ entirely or partially covered by K; further, after ordering the $h^{(\nu)}$ for all values of ν in a single fundamental sequence[9] $\theta', \theta'', \theta''', \ldots$, to be called F, by a

number possessing f is not known to be noncontradictory either." (*Brouwer 1952*, p. 141; see also *1928a*, p. 161.)

The *critical number* of a fleeing property is apparently what Brouwer (*1929a*, p. 161) calls the *Lösungszahl* of the property, that is, the (hypothetical) least natural number that possesses the property.]]

[5] [["We say that [[a number core]] a lies *apart* from [[a number core]] b if there is some natural number n such that $|b - a| > 2^{-n}$." (*Brouwer 1953*, p. 4.) See also below, p. 462, footnote 10a.]]

[6] [[The definition of "completely differentiable" requires too many preliminary definitions to be reproduced here; we refer the reader to the passage indicated in the text.]]

[7] A similar disappearance of a counterexample, due to the disappearance of the absence of a requisite algorithm, belongs to the realm of possibilities when one considers $\psi(x)$ *simply for a fixed fleeing property f*.

[[The *classical continuum* is the species of predeterminate intuitionistic real numbers; see *Brouwer 1952*, p. 142, bottom half of first column, and p. 143, top of first column, as well as above, p. 342, V.]]

[8] [[For the definition of "$\lambda^{(\nu)}$-interval" see below, p. 457.]]

[9] [[For the definition of "fundamental sequence" see below, p. 455.]]

unitary standard number we understand an infinitely proceeding sequence[10] $\theta^{(c_1)}$, $\theta^{(c_2)}$, $\theta^{(c_3)}, \ldots$ in which, for every ν, $\theta^{(c_\nu)}$ is an $h^{(\nu)}$ and $\theta^{(c_{\nu+1})}$ consists entirely of inner points of $\theta^{(c_\nu)}$. Then, the species of unitary standard numbers is identical with the species of accretion sequences[11] of a *dressed fan* w,[12] of which we can say—because every unitary number core, that is, every number core of K, coincides with a unitary standard number—that it *represents* K.

As a function of a variable number core x, either of K_r or of K, $g(x)$ is now obtained as follows.[13] Let f be a fleeing property; let $\kappa_1(f)$ be its critical number; let p_ν and q_ν be respectively the least and the greatest endcores of $\theta^{(\nu)}$; and let $\varphi_\nu(x)$ be the continuous function of K_r, or of K, that for the part of $\theta^{(\nu)}$ that belongs to K_r, or to K, is equal to

$$\frac{q_\nu - p_\nu}{2\pi} \sin 2\pi \frac{x - p_\nu}{q_\nu - p_\nu}$$

and, for $x \leqq p_\nu$ as well as for $x \geqq q_\nu$, is equal to 0. Then we put $g_\nu(x) \equiv x$ for $\nu = 1$, $g_\nu(x) \equiv \varphi_\nu(x)$ for $\nu = \kappa_1(f)$, and $g_\nu(x) \equiv 0$ for all other values of ν. Finally, we put

$$g(x) \equiv \sum_{\nu=1}^{\infty} g_\nu(x).$$

If we call a $\theta^{(\nu)}$ for which $\nu = \kappa_1(f)$ the *critical interval* of f and if we represent this by $i(f)$, then (at least for the current examples of $\digamma$ and f) not a single indication is at hand concerning the position of a possible $i(f)$; therefore, it seems at the outset that for every x every possibility of obtaining a guarantee for the nonbelonging to $i(f)$ is lacking, and so is for every unitary finite binary fraction[14] x every possibility of computing a ratio 1/3 for the lengths of the segments into which it would have to divide a possible $i(f)$ to which it would belong; therefore finally it seems that for every x every possibility of computing an ordinary derivative is lacking.

§ 3

This situation, however, changes when one intends to make the infinitely proceeding process of *the creation, by free choices, of a unitary standard number u* run parallel to the infinitely proceeding process of the successive judgments of the assignment of f to the successive natural numbers and moreover to take care that the creation process of u continually lags sufficiently far behind the process of judging that was just mentioned to prevent contact with an $i(f)$ that might possibly appear, so that there must come into existence a number core x of K for which $g(x)$ possesses an ordinary derivative equal to 1.

Once this insight has been obtained, it is not far-fetched to observe that the way, indicated here, in which u comes to exist is at hand for all the accretion sequences

[10] [[See the definition of "unbounded choice sequence" below, p. 446; "infinitely proceeding sequence" was used in *Brouwer 1952*, p. 142, bottom of first column; see also 3.1.1. in *Heyting 1956*, pp. 32–34.]]

[11] [["Accretion sequence" ("accretiereeks") is here apparently used for "infinitely proceeding sequence in a dressed spread".]]

[12] [[For the definition of "dressed fan" see *Brouwer 1953*, p. 16, first paragraph.]]

[13] [[The remark made in footnote 3 applies to the function $g(x)$.]]

[14] [[For the definition of "finite binary fraction" see below, p. 457, footnote 1.]]

of the elements of a subfan w' of w that is obtained from w by the deletion, from the species of constituents[15] that are admitted for the nodes of w, of a possible $i(f)$, as well of the two λ-intervals that are of the same length as $i(f)$ and are partially covered by $i(f)$. Therefore, for every number core x of K that is represented by this dressed fan w', $g(x)$ possesses an ordinary derivative.

By means of the same fan w' it is even possible to exhibit, for every natural number n, a measurable core species S_n that is contained in K, has a content greater than $1 - 2^{-4n}$, and in which $g(x)$ everywhere possesses an ordinary derivative.[16] For that one establishes first of all for every n one of the following facts: either for $\nu \leqq n$ no critical interval of f occurs among the $h^{(\nu)}$ or for some $m \leqq n$ a critical interval of f occurs among the $h^{(m)}$. Further, there is chosen for S_n, in the first case, the core species of K represented by w' and, in the second case, the species of the cores of K that lie apart from the two endcores of $i(f)$. If we further observe that the union of the infinitely proceeding sequence of the S_ν forms a measurable core species that is contained in K and has content 1, then $g(x)$ turns out to be a *continuous, monotonic, real, full function of K that is differentiable almost everywhere.*

And since the predeterminate elements of w' represent number cores of K_r, K_r also possesses an (everywhere dense, ever unfinished, and ever enumerable) core species in which $g(x)$ is everywhere differentiable.

[15] [[For the definition of "constituent" see *Brouwer 1953*, p. 7.]]

[16] [[For the definitions of "measurable core species" and "content" see *Brouwer 1919*, pp. 26–33; see also *Heyting 1956*, chap. V, secs. 1 and 3.]]