The principles of arithmetic, presented by a new method

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(*1889*)

Written in Latin, this small book was Peano's first attempt at an axiomatization of mathematics in a symbolic language. Peano had already (*1888*) used the logic of Boole and Schröder in mathematical investigations and introduced into it a number of innovations that marked a definite advance upon the work of his predecessors: for instance, the use of different signs for logical and mathematical operations, and a distinction between categorical and conditional propositions that was to lead him to quantification theory (these were innovations relatively to Boole and Schröder—not to Frege, whose work Peano did not know at that time). In the present work, after having introduced logical notions and formulas, Peano undertakes to rewrite arithmetic in symbolic notation. But he aspires to more than that: the book deals also with fractions, real numbers, even the notion of limit and definitions in point-set theory.

The initial arithmetic notions are "number", "one", "successor", and "is equal to", and nine axioms are stated concerning these notions. Today we would consider that Axioms 2, 3, 4, and 5, which deal with identity, belong to the underlying logic. This leaves the five axioms that have become universally known as "the Peano axioms". The last one, Axiom 9, is the translation of the principle of mathematical induction. It is formulated in terms of classes, and it contains a class variable, "k" (it even involves the class of all classes, K). Peano acknowledges (*1891b*, p. 93) that his axioms come from Dedekind (*1888*, art. 71, definition of a simply infinite system; see also below, pp. 100–101). As for Frege, Peano learned of his work immediately after the publication of *Arithmetices principia*.[a]

From the outset, Peano uses the notation $x + 1$ for the successor function. He then introduces addition (§ 1, 18) and multiplication (§ 4, 1 and 2) as "definitions". These definitions are recursive definitions, although Peano does not have at his disposal in his system anything like Dedekind's powerful Theorem 126 (*1888*), which justifies such definitions. Peano does not explicitly claim that these definitions are eliminable, but, just as he does for ordinary definitions (that of subtraction, for example), he puts them under the heading "Definition", although they do not satisfy his own statement on that score (p. 93), namely, that the right side of a definitional equation is "an aggregate of

[a] In the list of references appended to *Arithmetices principia* (below, p. 86, footnote 1) Frege's name does not occur; but Peano mentions and even quotes Frege in his very next paper on logic (*1891*).

signs having a known meaning". He proves for addition a theorem (§ 1, 19) stating that "for every a and b, $a, b \varepsilon N .\text{Ɔ}. a + b \varepsilon N$", and a similar theorem (§ 4, 3) for multiplication; but these theorems are far from having the same effect as Dedekind's Theorem 126.

The ease with which we read Peano's booklet today shows how much of his notation has found its way, either directly or in a somewhat modified form, into contemporary logic. ε is there, with the distinction between elementhood and subclasshood (except for classes of one element—see formula 56 in Part IV, p. 90 below; for such classes the distinction will appear the following year (*1890*, p. 192)). The inverted C, Ɔ, will become ⊃.

The logical part of the work presents formulas of the propositional calculus, of the calculus of classes, and a few of quantification theory. Peano's notation is quite superior to that of Boole and Schröder, and it marks an important transition toward modern logic. Some distinction is made between the calculus of propositions and that of classes (ε, for example, already introduces an asymmetry between propositions and classes); we now have two different calculi, not just two interpretations of the same calculus. The notation for the universal quantifier is new and convenient. There is, however, a grave defect. The formulas are simply listed, not derived; and they could not be derived, because no rules of inference are given. Peano introduces a notation for substitution (V 4, p. 91) but does not state any rule. What is far more important, he does not have any rule that would play the role of the rule of detachment. The result is that, for all his meticulousness in the writing of formulas, he has no logic that he can use. The point is vividly illustrated by the first proof he gives, that of

11. $2 \varepsilon N$

(below, p. 94). What is presented as a proof is actually a list of formulas that are such that, from the point of view of the working mathematician, each one is very close to the next. But, however close two successive formulas may be, the logician cannot pass from one to the next because of the absence of rules of inference. The proof does not get off the ground.

In the proof just mentioned (and it is typical of Peano's proofs), the passage from formulas (1) and (2) to formula (3) cannot be carried out by a *formal* procedure; it requires some intuitive logical argument, which the reader has to supply. The proof brings out the whole difference between an axiomatization, even written in symbols and however careful it may be, and a formalization. The absence of a rule of detachment in Peano's booklet (and other works) is apparently connected with his inadequate interpretation of the conditional. He reads "a Ɔ b" as "from a one deduces b" ("ab a deducitur b"), which remains vague; truth values are not used at all in the work below, and only marginally in Peano's subsequent writings.

In a series of papers (*1891*, *1891a*, *1891b*; see also *1890*) that form a sequel to *Arithmetices principia* and a transition toward the first volume of the *Formulaire* (*1895*), Peano undertakes to prove the logical formulas that he simply listed in the logical part of the work below. Just like his arithmetic proofs, his logical proofs suffer from the absence of rules of inference. In the proof of proposition 9 (*1891a*, p. 27), for example, he strings conditionals one after another; when ultimately he does detach, it is by a move totally unjustified in his system. For a pertinent critique of that aspect of Peano's work see *Frege 1896* and *1896a*. Some of Peano's explanations tend to suggest that his logical laws should perhaps be taken as rules of inference, not as formulas in a logical language; this, however, would not yield a coherent interpretation of his system.

In the work below and in the various editions of the *Formulaire* that were to follow, Peano intends to cover much more ground than Frege does in his *Begriffsschrift* and his subsequent works, but he does not till that ground to any depth comparable to what Frege does in his self-alloted field. Peano's writings, of minor significance for logic proper, showed how mathematical theories can be expressed in one symbolic language. These writings rapidly gained a wide influence and greatly contributed to the diffusion of the new ideas.

Arithmetices principia consists of a long explanatory preface and ten sections: § 1 Number and addition, § 2 Subtraction, § 3 Maxima and minima, § 4 Multiplication, § 5 Powers, § 6 Division, § 7 Various theorems, § 8 Ratios of numbers, § 9 Systems of rationals, irrationals, § 10 Systems of quantities. Below we print the preface and § 1 *in extenso*; from §§ 2, 4, 5, and 6 we give the "explanations" and "definitions", omitting the theorems, and we leave out the other sections entirely. The omitted parts consist almost exclusively of formulas and are readily available in Peano's collected works (*1958*).

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Preface

Questions that pertain to the foundations of mathematics, although treated by many in recent times, still lack a satisfactory solution. The difficulty has its main source in the ambiguity of language.

That is why it is of the utmost importance to examine attentively the very words we use. My goal has been to undertake this examination, and in this paper I am presenting the results of my study, as well as some applications to arithmetic.

I have denoted by signs all ideas that occur in the principles of arithmetic, so that every proposition is stated only by means of these signs.

The signs belong either to logic or to arithmetic proper. The signs of logic that occur here are ten in number, although not all are necessary. In the first part of the present paper ⟦Logical notations⟧ the use of these signs, as well as some of their properties, is explained in ordinary language. It was not my intention to present their theory more fully there. The signs of arithmetic are explained wherever they occur.

With these notations, every proposition assumes the form and the precision that equations have in algebra; from the propositions thus written other propositions are deduced, and in fact by procedures that are similar to those used in solving equations. This is the main point of the whole paper.

Thus, having introduced the signs with which I can write the propositions of arithmetic, I have, in dealing with these propositions, used a method that, because it will have to be followed in other studies too, I shall present briefly here.

Among the signs of arithmetic, those that can be expressed by other signs of arithmetic together with the signs of logic represent the ideas that we can define. Thus, I have defined all signs except the four that are contained in the explanations of § 1. If, as I think, these cannot be reduced any further, it is not possible to define the ideas expressed by them through ideas assumed to be known previously.

Propositions that are deduced from others by the operations of logic are *theorems*; propositions that are not thus deduced I have called *axioms*. There are nine of these axioms (§ 1), and they express the fundamental properties of the signs that lack definition.

In §§ 1–6 I have proved the ordinary properties of numbers. For the sake of brevity I have omitted proofs that are similar to other proofs given before. In order to express proofs with the signs of logic, the ordinary form of these proofs has to be changed; this transformation is sometimes rather difficult, but it is by means of it that the nature of the proof reveals itself most clearly.

In subsequent sections I deal with various subjects, so that the power of the method will be more apparent.

§ 7 contains a few theorems that pertain to number theory. In §§ 8–9 the definitions of rational and irrational numbers are found.

Finally, in § 10 I present a few theorems, which, I think, are new, pertaining to the theory of those objects that Cantor has called *Punktmengen* (*ensembles de points*).

In the present paper I have made use of the studies of other writers. The logical symbols and propositions contained in parts II, III, and IV, except for a few, are to be traced to the works of many writers, especially Boole.[1]

I introduced the sign ε, which should not be confused with the sign Ɔ; I also introduced applications of inversion in logic, as well as a few other conventions, in order to be able to express any proposition whatsoever.

For proofs in arithmetic, I used *Grassmann 1861*.

The recent work of Dedekind (*1888*) was also most useful to me; in it, questions pertaining to the foundations of numbers are acutely examined.

This little book of mine is intended to give an example of the new method. With these notations we can state and prove innumerable other propositions, whether they pertain to rational or to irrational numbers. But to deal with other theories new signs denoting new objects must be introduced. However, I think that the propositions of any science can be expressed by these signs of logic alone, provided we add signs representing the objects of that science.

Logical notations

I. *Punctuation*

The letters a, $b, \ldots, x, y, \ldots, x', y', \ldots$ denote indeterminate objects. We denote well-determined objects by signs or by the letters P, K, N, ...

We shall generally write signs on a single line. To show the order in which they should be taken, we use *parentheses*, as in algebra, or *dots*, ., :, .:., ::, and so on.

To understand a formula divided by dots we first take together the signs that are not separated by any dot, next those separated by one dot, then those separated by two dots, and so on.

For example, let $a, b, c, \ldots$ be any signs. Then $ab.cd$ means $(ab)(cd)$; and $ab.cd\!:\!ef.gh\,.\!:\,k$ means $(((ab)(cd))((ef)(gh)))k$.

Punctuation signs may be omitted if formulas differing in punctuation have the

[1] See *Boole 1847*, *1848*, *1854*, and *Schröder 1877*. Schröder had already dealt with some questions relevant to logic in an earlier work (*1873*). I presented the theories of Boole and Schröder very briefly in a book of mine (*1888*). See also *Peirce 1880*, *1885*, *Jevons 1883*, *MacColl 1877*, *1878*, *1878a*, and *1880*.

same meaning or if only one formula, which is just the one we want to write, has meaning.

To avoid the danger of ambiguity we never use . or : as signs for arithmetic operations.

The only kind of parentheses is ( ); if parentheses and dots occur in the same formula, what is contained within parentheses is taken together first.

II. *Propositions*

The sign P means *proposition.*

The sign $\cap$ is read *and.* Let a and b be propositions; then $a \cap b$ is the simultaneous affirmation of the propositions a and b. For the sake of brevity, we ordinarily write ab instead of $a \cap b$.

The sign $-$ is read *not.* Let a be a P; then $-a$ is the negation of the proposition a.

The sign $\cup$ is read *or* ⟦*vel*⟧. Let a and b be propositions; then $a \cup b$ is the same as $-:-a.-b$.

[The sign V means *the true*, or *identity*; but we never use this sign.]

The sign Λ means *the false*, or *the absurd.*

[The sign C means *is a consequence of*; thus $b \mathrm{C} a$ is read b *is a consequence of the proposition* a. But we never use this sign.]

The sign Ɔ means *one deduces* ⟦*deducitur*⟧;[2] thus $a \supset b$ means the same as $b \mathrm{C} a$. If propositions a and b contain the indeterminate objects $x, y, \ldots$, that is, are conditions between these objects, then $a \supset_{x,y,\ldots} b$ means: whatever $x, y, \ldots$ may be, from the proposition a one deduces b. If there is no danger of any ambiguity, we write only $\supset$ instead of $\supset_{x,y,\ldots}$.

The sign = means *is equal to* ⟦*est aequalis*⟧. Let a and b be propositions; then $a = b$ means the same as $a \supset b . b \supset a$; the proposition $a =_{x,y,\ldots} b$ means the same as $a \supset_{x,y,\ldots} b . b \supset_{x,y,\ldots} a$.

III. *Propositions of logic*

Let $a, b, c, \ldots$ be propositions. Then we have:

1. $a \supset a$.
2. $a \supset b . b \supset c :\supset. a \supset c$.
3. $a = b .=: a \supset b . b \supset a$.
4. $a = a$.
5. $a = b .=. b = a$.
6. $a = b . b \supset c :\supset. a \supset c$.
7. $a \supset b . b = c :\supset. a \supset c$.
8. $a = b . b = c :\supset. a = c$.
9. $a = b .\supset. a \supset b$.

[2] ⟦Peano reads $a \supset b$ "ab a deducitur b". Translated word for word, this either would be awkward ("from a is deduced b") or would reverse the relative positions of a and b ("b is deduced from a"), which would lead to misinterpretations when the sign is read alone. Peano himself uses "on déduit" for "deducitur" when writing in French (for instance, *1890*, p. 184), and this led to the translation adopted here.⟧

10. $a = b .\supset. b \supset a.$

11. $ab \supset a.$
12. $ab = ba.$
13. $a(bc) = (ab)c = abc.$
14. $aa = a.$
15. $a = b .\supset. ac = bc.$
16. $a \supset b .\supset. ac \supset bc.$
17. $a \supset b . c \supset d :\supset. ac \supset bd.$
18. $a \supset b . a \supset c :=. a \supset bc.$
19. $a = b . c = d :\supset. ac = bd.$

20. $-(-a) = a.$
21. $a = b .=. -a = -b.$
22. $a \supset b .=. -b \supset -a.$

23. $a \cup b .= \therefore - : -a . -b.$
24. $-(ab) = (-a) \cup (-b).$
25. $-(a \cup b) = (-a)(-b).$
26. $a \supset . a \cup b.$
27. $a \cup b = b \cup a.$
28. $a \cup (b \cup c) = (a \cup b) \cup c = a \cup b \cup c.$
29. $a \cup a = a.$
30. $a(b \cup c) = ab \cup ac.$
31. $a = b .\supset. a \cup c = b \cup c.$
32. $a \supset b .\supset. a \cup c \supset b \cup c.$
33. $a \supset b . c \supset d :\supset: a \cup c .\supset. b \cup d.$
34. $b \supset a . c \supset a :=. b \cup c \supset a.$

35. $a - a = \Lambda.$
36. $a\Lambda = \Lambda.$
37. $a \cup \Lambda = a.$
38. $a \supset \Lambda .=. a = \Lambda.$
39. $a \supset b .=. a - b = \Lambda.$
40. $\Lambda \supset a.$
41. $a \cup b = \Lambda .=: a = \Lambda . b = \Lambda.$

42. $a \supset . b \supset c :=: ab \supset c.$
43. $a \supset . b = c :=. ab = ac.$

Let α be a relation sign (for example, $=, \supset$), so that $a \, \alpha \, b$ is a proposition. Then, instead of $-. a \, \alpha \, b$, we write $a -\alpha\, b$; that is,

$$a -= b .=: -. a = b.$$

$$a -\supset b .=: -. a \supset b.$$

Thus the sign $-=$ means *is not equal to*. If proposition a contains the indeterminate

x, then $a -\!=_x \Lambda$ means: there are x that satisfy condition a. The sign $-\!\supset$ means *one does not deduce.*

Similarly, if α and β are relation signs, instead of $a\ \alpha\ b\,.\,a\ \beta\ b$ and $a\ \alpha\ b\ .\cup.\ a\ \beta\ b$, we can write $a\ .\alpha\beta.\ b$ and $a\ .\alpha \cup \beta.\ b$, respectively. Thus, if a and b are propositions, the formula $a\ .\supset\!-\!=.\ b$ says: from a one deduces b, but not conversely.

$$a\ .\supset\!-\!=.\ b\ :=:\ a \supset b\,.\,b -\!\supset a.$$

We have the formulas:

$$a \supset b\,.\,b \supset c\,.\,a -\!\supset c := \Lambda.$$
$$a = b\,.\,b = c\,.\,a -\!= c := \Lambda.$$
$$a \supset b\,.\,b \supset\!-\!= c\ :\supset.\ a \supset\!-\!= c.$$
$$a \supset\!-\!= b\,.\,b \supset c\ :\supset.\ a \supset\!-\!= c.$$

But we shall rarely use these devices.

IV. *Classes*

The sign K means *class*, or aggregate of objects.

The sign ε means *is*. Thus $a\ \varepsilon\ b$ is read *a is a b*; $a\ \varepsilon\ \mathrm{K}$ means *a is a class*; $a\ \varepsilon\ \mathrm{P}$ means *a is a proposition.*

Instead of $-(a\ \varepsilon\ b)$ we write $a -\!\varepsilon\ b$; the sign $-\varepsilon$ means *is not*; that is,

44. $a -\!\varepsilon\ b\ .=:\ -.a\ \varepsilon\ b.$

The sign $a, b, c\ \varepsilon\ m$ means: a, b, and c are m; that is,

45. $a, b, c\ \varepsilon\ m\ .=:\ a\ \varepsilon\ m\,.\,b\ \varepsilon\ m\,.\,c\ \varepsilon\ m.$

Let a be a class; then $-a$ means the class composed of the individuals that are not a.

46. $a\ \varepsilon\ \mathrm{K}\ .\supset:\ x\ \varepsilon\ -a\ .=.\ x -\!\varepsilon\ a.$

Let a and b be classes; $a \cap b$, or ab, is the class composed of the individuals that are at the same time a and b; $a \cup b$ is the class composed of the individuals that are a or b.

47. $a, b\ \varepsilon\ \mathrm{K}\ .\supset\therefore\ x\ \varepsilon.\ ab\ :=:\ x\ \varepsilon\ a\,.\,x\ \varepsilon\ b.$
48. $a, b\ \varepsilon\ \mathrm{K}\ .\supset\therefore\ x\ \varepsilon.\ a \cup b\ :=:\ x\ \varepsilon\ a\ .\cup.\ x\ \varepsilon\ b.$

The sign Λ denotes the class that contains no individuals. Thus,

49. $a\ \varepsilon\ \mathrm{K}\ .\supset\therefore\ a = \Lambda\ :=:\ x\ \varepsilon\ a\ .=_x \Lambda.$

[We shall not use the sign V, which denotes the class composed of all individuals under consideration.]

The sign $\supset$ means *is contained in*. Thus $a \supset b$ means *class a is contained in class b.*

50. $a, b\ \varepsilon\ \mathrm{K}\ .\supset\therefore\ a \supset b\ :=:\ x\ \varepsilon\ a\ .\supset_x.\ x\ \varepsilon\ b.$

[The formula b C a could mean *class b contains class a*; but we shall not use the sign C.]

The signs Λ and $\supset$ here have a meaning that differs somewhat from the meaning given above; but no ambiguity will arise. If we are dealing with propositions, these signs are read *the absurd* and *one deduces*; but, if we are dealing with classes, they are read *nothing* and *is contained in.*

If a and b are classes, the formula $a = b$ means $a \supset b . b \supset a$. Thus,

51. $a, b \,\varepsilon\, \mathrm{K} \,.\supset\therefore a = b :=: x \,\varepsilon\, a \,.=_x.\, x \,\varepsilon\, b.$

Propositions 1–41 still hold if $a, b, \ldots$ denote classes; in addition, we have:

52. $a \,\varepsilon\, b \,.\supset.\, b \,\varepsilon\, \mathrm{K}.$
53. $a \,\varepsilon\, b \,.\supset.\, b -= \Lambda.$
54. $a \,\varepsilon\, b . b = c :\supset.\, a \,\varepsilon\, c.$
55. $a \,\varepsilon\, b . b \supset c :\supset.\, a \,\varepsilon\, c.$

Let s be a class and k a class contained in s; then we say that k is an individual of class s if k consists of just one individual. Thus,

56. $s \,\varepsilon\, \mathrm{K} . k \supset s :\supset:: k \,\varepsilon\, s \,.=\therefore k -= \Lambda : x, y \,\varepsilon\, k \,.\supset_{x,y}.\, x = y.$

V. *Inversion*

The sign of inversion is [ ], and we shall explain its use in Part VI. Here we simply present some special cases.

1. Let a be a proposition containing the indeterminate x; then the expression $[x \,\varepsilon]\, a$, which is read *the x for which a*, or *the solutions* of the condition a, or its *roots*, means the class composed of the individuals that satisfy condition a. Thus,

57. $a \,\varepsilon\, \mathrm{P} \,.\supset: [x \,\varepsilon]\, a \,.\varepsilon\, \mathrm{K}.$
58. $a \,\varepsilon\, \mathrm{K} \,.\supset\therefore [x \,\varepsilon].\, x \,\varepsilon\, a := a.$
59. $a \,\varepsilon\, \mathrm{P} \,.\supset\therefore x \,\varepsilon.\, [x \,\varepsilon]\, a := a.$

Let α and β be propositions containing the indeterminate x; we have:

60. $[x \,\varepsilon]\, (\alpha\beta) = ([x \,\varepsilon]\, \alpha)([x \,\varepsilon]\, \beta).$
61. $[x \,\varepsilon] -\alpha = -[x \,\varepsilon]\, \alpha.$
62. $[x \,\varepsilon]\, (\alpha \cup \beta) = [x \,\varepsilon]\, \alpha \cup [x \,\varepsilon]\, \beta.$
63. $\alpha \supset_x \beta \,.=.\, [x \,\varepsilon]\, \alpha \supset [x \,\varepsilon]\, \beta.$
64. $a =_x \beta \,.=.\, [x \,\varepsilon]\, \alpha = [x \,\varepsilon]\, \beta.$

2. Let x and y be any objects whatsoever; we consider as a new object the system composed of the object x and the object y, and we denote it by the sign (x, y); and similarly if we have a greater number of objects. Let α be a proposition containing the indeterminates x and y; then $[(x, y) \,\varepsilon]\, \alpha$ means the class composed of the objects (x, y) that satisfy the condition α. We have:

65. $\alpha \supset_{x,y} \beta \,.=.\, [(x, y) \,\varepsilon]\, \alpha \supset [(x, y) \,\varepsilon]\, \beta.$
66. $[(x, y) \,\varepsilon]\, \alpha -= \Lambda \,.=\therefore [x \,\varepsilon].\, [y \,\varepsilon]\, \alpha -= \Lambda :-= \Lambda.$

3. Let $x \,\alpha\, y$ be a relation between the indeterminates x and y (for example, in logic, the relations $x = y$, $x -= y$, $x \supset y$; in arithmetic, $x < y$, $x > y$, and so on). Then the sign $[\varepsilon]\, \alpha\, y$ denotes the x that satisfy the relation $x \,\alpha\, y$. For the sake of convenience, we use the sign $\ni$ instead of the sign $[\varepsilon]$. Thus, $\ni \alpha\, y \,.=: [x \,\varepsilon].\, x \,\alpha\, y$, and the sign $\ni$ is read *the objects that*. For example, let y be a number; then $\ni < y$ denotes the class formed by the numbers x that satisfy the condition $x < y$, that is, *the objects that are smaller than y*, or simply *the objects smaller than y*. Similarly, if the sign D means *divides* or *is a*

divisor of, the formula $\ni D$ means *the objects that divide* or *the divisors*. It follows that $x \mathrel{\varepsilon} \ni \alpha\, y = x\, \alpha\, y$.

4. Let α be a formula containing the indeterminate x. Then the expression $x'\,[x]\,\alpha$, which is read *x' being substituted for x in α*, denotes the formula obtained if, in α, we read x' for x. It follows that $x\,[x]\,\alpha = \alpha$.

5. Let α be a formula that contains the indeterminates $x, y, \ldots$. Then $(x', y', \ldots)[x, y, \ldots]\,\alpha$, which is read *$x', y' \ldots$ being substituted for $x, y, \ldots$ in α*, denotes the formula obtained if, in α, the letters $x_1', y', \ldots$ are written for $x, y, \ldots$. It follows that $(x, y)\,[x, y]\,\alpha = \alpha$.

VI. *Functions*

The symbols of logic introduced above suffice to express any proposition of arithmetic, and we shall use only these. We explain here briefly some other symbols that may be useful.

Let s be a class; we assume that equality is defined between the objects of the system s so as to satisfy the conditions:

$$a = a.$$
$$a = b \,.\!=\!.\, b = a.$$
$$a = b \,.\, b = c \;:\!\supset\!.\; a = c.$$

Let φ be a sign or an aggregate of signs such that, if x is an object of the class s, the expression φx denotes a new object; we assume also that equality is defined between the objects φx; further, if x and y are objects of the class s and if $x = y$, we assume it is possible to deduce $\varphi x = \varphi y$. Then the sign φ is said to be a *function presign* [[*praesignum*]] *in the class s*, and we write $\varphi \mathrel{\varepsilon} \mathrm{F}`s$:

$$s \mathrel{\varepsilon} \mathrm{K} \,.\!\supset\!::\, \varphi \mathrel{\varepsilon} \mathrm{F}`s \,.\!=\!.\!\therefore\, x, y \mathrel{\varepsilon} s \,.\, x = y \;:\!\supset_{x,y}.\; \varphi x = \varphi y.$$

If, x being any object of the class s, the expression $x\varphi$ denotes a new object and $x\varphi = y\varphi$ follows from $x = y$, then we say that φ is a *function postsign* [[*postsignum*]] *in the class s*, and we write $\varphi \mathrel{\varepsilon} s'\mathrm{F}$:

$$s \mathrel{\varepsilon} \mathrm{K} \,.\!\supset\!::\, \varphi \mathrel{\varepsilon} s'\mathrm{F} \,.\!=\!.\!\therefore\, x, y \mathrel{\varepsilon} s \,.\, x = y \;:\!\supset_{x,y}.\; x\varphi = y\varphi.$$

Examples. Let a be a number; then $a+$ is a function presign in the class of numbers, and $+a$ is a function postsign; for any number x, formulas $a + x$ and $x + a$ denote new numbers; $a + x = a + y$ and $x + a = y + a$ follow from $x = y$. Thus,

$$a \mathrel{\varepsilon} \mathrm{N} \,.\!\supset\!:\, a+ \,.\varepsilon.\, \mathrm{F}`\mathrm{N}.$$
$$a \mathrel{\varepsilon} \mathrm{N} \,.\!\supset\!:\, +a \,.\varepsilon.\, \mathrm{N}'\mathrm{F}.$$

Let φ be a function presign in the class s. Then $[\varphi]\,y$ denotes the class composed of the x that satisfy the condition $\varphi x = y$; that is,

$$\textit{Def.}\quad s \mathrel{\varepsilon} \mathrm{K} \,.\, \varphi \mathrel{\varepsilon} \mathrm{F}`s \;:\!\supset\!:\; [\varphi]\,y \,.\!=\!.\, [x \mathrel{\varepsilon}]\,(\varphi x = y).$$

The class $[\varphi]\,y$ may contain one or several individuals, or none at all. We have

$$s \mathrel{\varepsilon} \mathrm{K} \,.\, \varphi \mathrel{\varepsilon} \mathrm{F}`s \;:\!\supset\!:\; y = \varphi x \,.\!=\!.\, x \mathrel{\varepsilon} [\varphi]\,y.$$

But if φy consists of just one individual, we have $y = \varphi x \,.\!=\!.\, x = [\varphi]\,y$.

Let φ be a function postsign; we write similarly:

$$s \varepsilon \mathrm{K} . \varphi \varepsilon s\text{'}\mathrm{F} :\supset .\cdot . \, y [\varphi] = [x \varepsilon] (x\varphi = y).$$

The sign [ ] is called the *inversion sign*, and we have already presented some of its uses in logic. If α is a proposition containing the indeterminate x and a is a class composed of the individuals x that satisfy the condition α, we have $x \varepsilon a .= \alpha$, and then $a = [x \varepsilon] \alpha$, as in V 1.

Let α be a formula containing the indeterminate x and let φ be a function presign that yields the formula α when written before the letter x; that is, let $\alpha = \varphi x$. Then we have $\varphi = \alpha [x]$, and, if x' is a new object, we have $\varphi x' = \alpha [x] x'$; that is, if α is a formula containing the indeterminate x, then $\alpha [x] x'$ means what is obtained when, in α, we put x' for x.

Similarly, let α be a formula containing the indeterminate x and let φ be a function postsign, such that $x\varphi = \alpha$; it follows that $\varphi = [x] \alpha$. Then, if x' is a new object, we have $x'\varphi = x' [x] \alpha$; that is, $x' [x] \alpha$ again denotes what is obtained when, in α, we read x' for x, as in V 4.

The sign [ ] can have other uses in logic, which we present only briefly, since we shall not use it in these ways. Let a and b be two classes; then $[a \cap] b$ (or $b [\cap a]$) denotes the classes x that satisfy the condition $b = a \cap x$ (or the condition $b = x \cap a$). If b is not contained in a, no class satisfies this condition; if b is contained in a, the sign $b [\cap a]$ denotes all classes that contain b and are contained in $b \cup -a$.

In arithmetic, let a and b be numbers; then $b [+ a]$ (or $[a +] b$) denotes the number x that satisfies the condition $b = x + a$ (or $b = a + x$), that is, $b - a$. Similarly we have $b [\times a] = [a \times] b = b/a$. This sign can even find a use in analysis; thus,

$$y = \sin x .=. \, x \varepsilon [\sin] y \qquad \text{(instead of } x = \text{arc sin } y)$$
$$d\mathrm{F}(x) = f(x)dx .=. \, \mathrm{F}(x) \varepsilon [d] f(x)dx \qquad \text{(instead of } \mathrm{F}(x) = \textstyle\int f(x)dx).$$

Let φ again be a function presign in a class s and let k be a class contained in s; then φk denotes the class consisting of all φx, where the x are the objects of class k; that is,

Def. $s \varepsilon \mathrm{K} . k \varepsilon \mathrm{K} . k \supset s . \varphi \varepsilon \mathrm{F}\text{‘}s :\supset . \, \varphi k = [y \varepsilon] (k . [\varphi] y :-= \Lambda)$,

or $s \varepsilon \mathrm{K} . k \varepsilon \mathrm{K} . k \supset s . \varphi \varepsilon \mathrm{F}\text{‘}s :\supset . \, \varphi k = [y \varepsilon] ([x \varepsilon] : x \varepsilon k . \varphi x = y .\cdot . -= \Lambda)$.

Def. $s \varepsilon \mathrm{K} . k \varepsilon \mathrm{K} . k \supset s . \varphi \varepsilon s\text{'}\mathrm{F} :\supset . \, k\varphi = [y \varepsilon] (k . y [\varphi] :-= \Lambda)$.

Thus, if $\varphi \varepsilon \mathrm{F}\text{‘}s$, then φs denotes the class composed of all φx, where the x are objects of the class s. We have:

$$s \varepsilon \mathrm{K} . \varphi \varepsilon \mathrm{F}\text{‘}s . y \varepsilon \varphi s :\supset : \varphi [\varphi] y = y.$$
$$s \varepsilon \mathrm{K} . a, b \varepsilon \mathrm{K} . a \supset s . b \supset s . \varphi \varepsilon \mathrm{F}\text{‘}s :\supset . \, \varphi(a \cup b) = (\varphi a) \cup (\varphi b).$$
$$s \varepsilon \mathrm{K} . \varphi \varepsilon \mathrm{F}\text{‘}s :\supset . \, \varphi\Lambda = \Lambda.$$
$$s \varepsilon \mathrm{K} . a, b \varepsilon \mathrm{K} . b \supset s . a \supset b . \varphi \varepsilon \mathrm{F}\text{‘}s :\supset . \, \varphi a \supset \varphi b.$$
$$s \varepsilon \mathrm{K} . a, b \varepsilon \mathrm{K} . a \supset s . b \supset s . \varphi \varepsilon \mathrm{F}\text{‘}s :\supset . \, \varphi(ab) \supset (\varphi a)(\varphi b).$$

Let a be a class; then $a \cap \mathrm{K}$ (or $\mathrm{K} \cap a$, or $\mathrm{K}a$) denotes all classes of the form $a \cap x$ (or $x \cap a$, or xa), where x is any class; that is, $\mathrm{K}a$ denotes the classes that are contained in a. The formula $x \varepsilon \mathrm{K}a$ means the same as $x \varepsilon \mathrm{K} . x \supset a$. We shall sometimes use this convention; thus KN means *a class of numbers*.

Similarly, if a is a class, $\mathrm{K} \cup a$ denotes the classes that contain a.

Let a be a number; then $a + \mathrm{N}$ (or $\mathrm{N} + a$) denotes *the numbers greater than the number* a; $a \times \mathrm{N}$ (or $\mathrm{N} \times a$ or $\mathrm{N}a$) denotes *the multiples of the number* a; a^{N} denotes *the powers of the number* a; N^2, $\mathrm{N}^3, \ldots$ denote *the squares, the cubes*, and so on.

Equality, product, and powers can be defined thus for function signs:

$$Def.\quad s\,\varepsilon\,\mathrm{K}\,.\,\varphi, \psi\,\varepsilon\,\mathrm{F}\text{‘}s \;:\!\mathrm{O}\!.\!\cdot.\; \varphi = \psi \;:=:\; x\,\varepsilon\,s\,.\mathrm{O}_x.\, \varphi x = \psi x.$$
$$Def.\quad s\,\varepsilon\,\mathrm{K}\,.\,\varphi\,\varepsilon\,\mathrm{F}\text{‘}s\,.\,\psi\,\varepsilon\,\mathrm{F}\text{‘}\varphi s\,.\,x\,\varepsilon\,s \;:\!\mathrm{O}.\; \psi\varphi x = \psi(\varphi x).$$

Thus, if we assume this definition, we have the new function presign $\psi\varphi$; it is called the *product of the signs* ψ and φ.

Similarly if φ and ψ are function postsigns.

The following proposition holds:

$$s\,\varepsilon\,\mathrm{K}\,.\,\varphi\,\varepsilon\,\mathrm{F}\text{‘}s\,.\,\varphi s\,\mathrm{O}\,s \;:\!\mathrm{O}\!:\; \varphi\varphi s\,\mathrm{O}\,s\,.\,\varphi\varphi\varphi s\,\mathrm{O}\,s\,.\text{and so on.}$$

The functions $\varphi\varphi$, $\varphi\varphi\varphi, \ldots$ are said to be *iterated* and are generally denoted by the signs φ^2, $\varphi^3, \ldots$, as powers of the operation φ.

But if φ is a function postsign, we can use the following more convenient notation without ambiguity:

$$Def.\quad s\,\varepsilon\,\mathrm{K}\,.\,\varphi\,\varepsilon\,s\text{'}\mathrm{F}\,.\,s\varphi\,\mathrm{O}\,s \;:\!\mathrm{O}\!:\; \varphi 1 = \varphi\,.\,\varphi 2 = \varphi\varphi\,.\,\varphi 3 = \varphi\varphi\varphi\,.\text{and so on.}$$

Assuming this definition, if $m, n\,\varepsilon\,\mathrm{N}$, we have $\varphi(m + n) = (\varphi m)(\varphi n)$; $(\varphi m)n = \varphi(mn)$.

If we use this definition in arithmetic, we obtain the following. We can denote the number that follows the number a by the more convenient sign $a+$; then $a + 1$, $a + 2, \ldots$, and, if b is a number, $a + b$, have the meaning of $a+$, $a++, \ldots$, which is clear from the definition in § 1 below. Proposition 6 in § 1 can be written $\mathrm{N} + \mathrm{O}\,\mathrm{N}$. If a, b, and c are numbers, then $a :+ b.c$ means $a + bc$, and $a :\times b.c$ means ab^c.

Function signs possess many other properties, especially if they satisfy the condition $\varphi x = \varphi y\,.\mathrm{O}.\, x = y$. A function sign that satisfies this condition is called *similar* by Dedekind (*ähnliche Abbildung*).[3]

But we lack the space to present these properties.

Remarks

A *definition*, or *Def.* for short, is a proposition of the form $x = a$ or $\alpha\,\mathrm{O}.\, x = a$, where a is an aggregate of signs having a known meaning, x is a sign or an aggregate of signs, hitherto without meaning, and α is the condition under which the definition is given.

A *theorem* (Theor. or Th.) is a proposition that is *proved*. If a theorem has the form $\alpha\,\mathrm{O}\,\beta$, where α and β are propositions, then α is called the *hypothesis* (Hyp. or, even shorter, Hp.) and β the *thesis* (Thes. or Ts.). Hyp. and Ts. depend on the form of the theorem; in fact, if we write $-\beta\,\mathrm{O}\,-\alpha$ instead of $\alpha\,\mathrm{O}\,\beta$, then $-\beta$ is the Hp. and $-\alpha$ the Ts.; if we write $\alpha{-}\beta = \Lambda$, Hp. and Ts. do not exist.

In any section below, the sign P followed by a number denotes the proposition indicated by that number in the same section. Propositions of logic are indicated by the sign L and the number of the proposition.

Formulas that do not fit on one line are continued on the next line without any intervening sign.

[3] [[Today "similar" has another meaning and instead we would say "equivalent".]]

§ 1. Numbers and addition

Explanations

The sign N means *number* (*positive integer*).

The sign 1 means *unity*.

The sign $a + 1$ means *the successor of* a, or a *plus* 1.

The sign $=$ means *is equal to*. We consider this sign as new, although it has the form of a sign of logic.

Axioms

1. $1 \,\varepsilon\, \mathrm{N}.$
2. $a \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, a = a.$
3. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!:\, a = b \,.\!=\!.\, b = a.$
4. $a, b, c \,\varepsilon\, \mathrm{N} \,.\!\supset\!\therefore\, a = b \,.\, b = c \,:\!\supset\!.\, a = c.$
5. $a = b \,.\, b \,\varepsilon\, \mathrm{N} \,:\!\supset\!.\, a \,\varepsilon\, \mathrm{N}.$
6. $a \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, a + 1 \,\varepsilon\, \mathrm{N}.$
7. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!:\, a = b \,.\!=\!.\, a + 1 = b + 1.$
8. $a \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, a + 1 \mathrel{-\!\!=} 1.$
9. $k \,\varepsilon\, \mathrm{K} \therefore 1 \,\varepsilon\, k \therefore x \,\varepsilon\, \mathrm{N} \,.\, x \,\varepsilon\, k \,:\!\supset_x\!.\, x + 1 \,\varepsilon\, k \,::\!\supset\!.\, \mathrm{N} \supset k.$

Definitions

10. $2 = 1 + 1;\ 3 = 2 + 1;\ 4 = 3 + 1;$ and so forth.

Theorems

11. $2 \,\varepsilon\, \mathrm{N}.$

Proof:

P 1 $.\!\supset\!:$	$1 \,\varepsilon\, \mathrm{N}$	(1)
$1\,[a]$ (P 6) $.\!\supset\!:$	$1 \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, 1 + 1 \,\varepsilon\, \mathrm{N}$	(2)
(1) (2) $.\!\supset\!:$	$1 + 1 \,\varepsilon\, \mathrm{N}$	(3)
P 10 $.\!\supset\!:$	$2 = 1 + 1$	(4)
$(4).(3).(2, 1 + 1)\,[a, b]$ (P 5) $:\!\supset\!:$	$2 \,\varepsilon\, \mathrm{N}$	(Theorem).

Note. We have written explicitly all the steps of this very easy proof. For the sake of brevity, we now write it as follows:

$$\mathrm{P}\,1 \,.\, 1\,[a]\,(\mathrm{P}\,6) \,:\!\supset\!:\, 1 + 1 \,\varepsilon\, \mathrm{N} \,.\, \mathrm{P}\,10 \,.\, (2, 1{+}1)\,[a, b]\,(\mathrm{P}\,5) \,:\!\supset\!:\, \mathrm{Th}.$$

or

$$\mathrm{P}\,1 \,.\, \mathrm{P}\,6 \,:\!\supset\!:\, 1 + 1 \,\varepsilon\, \mathrm{N} \,.\, \mathrm{P}\,10 \,.\, \mathrm{P}\,5 \,:\!\supset\!:\, \mathrm{Th}.$$

12. $3, 4, \ldots \,\varepsilon\, \mathrm{N}.$
13. $a, b, c, d \,\varepsilon\, \mathrm{N} \,.\, a = b \,.\, b = c \,.\, c = d \,:\!\supset\!:\, a = d.$

Proof: Hyp. P 4 $:\!\supset\!:\, a, c, d \,\varepsilon\, \mathrm{N} \,.\, a = c \,.\, c = d \,.\,$ P 4 $:\!\supset\!:$ Thes.

14. $a, b, c \,\varepsilon\, \mathrm{N} \,.\, a = b \,.\, b = c \,.\, a \mathrel{-\!\!=} c \,:=\, \Lambda.$

Proof: P 4 . L 39 $:\supset.$ Theor.

15. $a, b, c \varepsilon \mathrm{N} . a = b . b -\!= c :\supset. a -\!= c.$

16. $a, b \varepsilon \mathrm{N} . a = b :\supset. a + 1 = b + 1.$

16′. $a, b \varepsilon \mathrm{N} . a + 1 = b + 1 :\supset. a = b.$

Proof: P 7 = (P 16)(P 16′).

17. $a, b \varepsilon \mathrm{N} .\supset: a -\!= b .=. a + 1 -\!= b + 1.$

Proof: P 7 . L 21 $:\supset.$ Theor.

Definition

18. $a, b \varepsilon \mathrm{N} .\supset. a + (b + 1) = (a + b) + 1.$

Note. This definition has to be read as follows: if a and b are numbers, and if $(a + b) + 1$ has a meaning (that is, if $a + b$ is a number) but $a + (b + 1)$ has not yet been defined, then $a + (b + 1)$ means the number that follows $a + b$.

From this definition and also the preceding it follows that

$$a \varepsilon \mathrm{N} .\supset\therefore a + 2 = a + (1 + 1) = (a + 1) + 1,$$
$$a \varepsilon \mathrm{N} .\supset\therefore a + 3 = a + (2 + 1) = (a + 2) + 1,$$

and so forth.

Theorems

19. $a, b \varepsilon \mathrm{N} .\supset. a + b \varepsilon \mathrm{N}.$

Proof: $a \varepsilon \mathrm{N} . \mathrm{P}\,6 :\supset: a + 1 \varepsilon \mathrm{N} :\supset: 1 \varepsilon [b\, \varepsilon]\,\mathrm{Ts}.$ (1)

$a \varepsilon \mathrm{N} .\supset:: b \varepsilon \mathrm{N} . b \varepsilon [b\, \varepsilon]\,\mathrm{Ts} :\supset: a + b \varepsilon \mathrm{N} . \mathrm{P}\,6 :\supset: (a + b) + 1 \varepsilon \mathrm{N} . \mathrm{P}\,18 :\supset: a + (b + 1) \varepsilon \mathrm{N} :\supset: (b + 1) \varepsilon [b\, \varepsilon]\,\mathrm{Ts}.$ (2)

$a \varepsilon \mathrm{N} . (1) . (2) .\supset:: 1 \varepsilon [b\, \varepsilon]\,\mathrm{Ts} \therefore b \varepsilon \mathrm{N} . b \varepsilon [b\, \varepsilon]\,\mathrm{Ts} :\supset: b + 1 \varepsilon [b\, \varepsilon]\,\mathrm{Ts} \therefore ([b\, \varepsilon]\,\mathrm{Ts})\,[k]\,\mathrm{P}\,9 ::\supset: \mathrm{N} \supset [b\, \varepsilon]\,\mathrm{Ts}.\ (\mathrm{L}\,50) ::\supset: b \varepsilon \mathrm{N} .\supset \mathrm{Ts}.$ (3)

$(3) . (\mathrm{L}\,42) :\supset: a, b \varepsilon \mathrm{N} .\supset.$ Thesis. (Theor.).

20. *Def.* $a + b + c = (a + b) + c.$

21. $a, b, c \varepsilon \mathrm{N} .\supset. a + b + c \varepsilon \mathrm{N}.$

22. $a, b, c \varepsilon \mathrm{N} .\supset: a = b .=. a + c = b + c.$

Proof: $a, b \varepsilon \mathrm{N} . \mathrm{P}\,7 :\supset. 1 \varepsilon [c\, \varepsilon]\,\mathrm{Ts}.$ (1)

$a, b \varepsilon \mathrm{N} .\supset:: c \varepsilon \mathrm{N} . c \varepsilon [c\, \varepsilon]\,\mathrm{Ts} \therefore\supset\therefore a = b .=. a + c = b + c : a + c, b + c \varepsilon \mathrm{N} : a + c = b + c .=. a + c + 1 = b + c + 1 \therefore\supset\therefore a = b .=. a + (c + 1) = b + (c + 1) \therefore\supset\therefore (c + 1) \varepsilon [c\, \varepsilon]\,\mathrm{Ts}.$ (2)

$a, b \varepsilon \mathrm{N} . (1) . (2) :\supset:: 1 \varepsilon [c\, \varepsilon]\,\mathrm{Ts} \therefore c \varepsilon [c\, \varepsilon]\,\mathrm{Ts} .\supset. (c + 1) \varepsilon [c\, \varepsilon]\,\mathrm{Ts} ::\supset:: c \varepsilon \mathrm{N} .\supset. \mathrm{Ts}.$ (3)

(3) $\supset$ Theor.

23. $a, b, c \varepsilon \mathrm{N} .\supset. a + (b + c) = a + b + c.$

Proof: $a, b \varepsilon \mathrm{N} . \mathrm{P}\,18 . \mathrm{P}\,20 :\supset. 1 \varepsilon [c\, \varepsilon]\,\mathrm{Ts}.$ (1)

$a, b \varepsilon \mathrm{N} .\supset\therefore c \varepsilon \mathrm{N} . c \varepsilon [c\, \varepsilon]\,\mathrm{Ts} :\supset: a + (b + c) = a + b + c . \mathrm{P}\,7 :\supset: a + (b + c) + 1 = a + b + c + 1 . \mathrm{P}\,18 :\supset: a + (b + (c + 1)) = a + b + (c + 1) :\supset. c + 1 \varepsilon [c\, \varepsilon]\,\mathrm{Ts}.$ (2)

(1) (2) (P 9) $. \supset .$ Theor.

24. $a \,\varepsilon\, \mathrm{N} \,. \supset .\, 1 + a = a + 1.$

Proof: $\mathrm{P}\, 2 \,. \supset .\, 1 \,\varepsilon\, [a \,\varepsilon]\, \mathrm{Ts}.$ (1)

$a \,\varepsilon\, \mathrm{N} \,.\, a \,\varepsilon\, [a \,\varepsilon]\, \mathrm{Ts} \,: \supset :\, 1 + a = a + 1 \,: \supset :\, 1 + (a + 1) = (a + 1) + 1 \,: \supset :\, (a + 1) \,\varepsilon\, [a \,\varepsilon]\, \mathrm{Ts}.$ (2)

(1) (2) $. \supset .$ Theor.

24′. $a, b \,\varepsilon\, \mathrm{N} \,. \supset .\, 1 + a + b = a + 1 + b.$

Proof: Hyp. P 24 $: \supset :\, 1 + a = a + 1 \,.\,$ P 22 $: \supset .$ Thesis.

25. $a, b \,\varepsilon\, \mathrm{N} \,. \supset .\, a + b = b + a.$

Proof: $a \,\varepsilon\, \mathrm{N} \,.\, \mathrm{P}\, 24 \,: \supset :\, 1 \,\varepsilon\, [b \,\varepsilon]\, \mathrm{Ts}.$ (1)

$a \,\varepsilon\, \mathrm{N} \,. \supset .\!\cdot.\, b \,\varepsilon\, \mathrm{N} . b \,\varepsilon\, [b \,\varepsilon]\, \mathrm{Ts} \,: \supset :\, a + b = b + a \,.\, \mathrm{P}\, 7 \,: \supset :\, (a + b) + 1 = (b + a) + 1 \,.\, (a + b) + 1 = a + (b + 1) \,.\, (b + a) + 1 = 1 + (b + a) \,.\, 1 + (b + a) = (1 + b) + a \,.\, (1 + b) + a = (b + 1) + a \,: \supset :\, a + (b + 1) = (b + 1) + a \,: \supset :\, (b + 1) \,\varepsilon\, [b \,\varepsilon]\, \mathrm{Ts}.$ (2)

(1) (2) $. \supset .$ Theor.

26. $a, b, c \,\varepsilon\, \mathrm{N} \,. \supset :\, a = b \,. = .\, c + a = c + b.$

27. $a, b, c \,\varepsilon\, \mathrm{N} \,. \supset :\, a + b + c = a + c + b.$

28. $a, b, c, d \,\varepsilon\, \mathrm{N} \,.\, a = b \,.\, c = d \,: \supset .\, a + c = b + d.$

§ 2. Subtraction

Explanations

The sign $-$ is read *minus*.
The sign $<$ is read *is less than*.
The sign $>$ is read *is greater than*.

Definitions

1. $a, b \,\varepsilon\, \mathrm{N} \,. \supset :\, b - a = \mathrm{N}\, [x \,\varepsilon]\, (x + a = b).$
2. $a, b \,\varepsilon\, \mathrm{N} \,. \supset :\, a < b \,. = .\, b - a \,-\!\!= \Lambda.$
3. $a, b \,\varepsilon\, \mathrm{N} \,. \supset :\, b > a \,. = .\, a < b.$

$a + b - c = (a + b) - c\,;\; a - b + c = (a - b) + c\,;\; a - b - c = (a - b) - c.$

§ 4. Multiplication

Definitions

1. $a \,\varepsilon\, \mathrm{N} \,. \supset .\, a \times 1 = a.$
2. $a, b \,\varepsilon\, \mathrm{N} \,. \supset .\, a \times (b + 1) = a \times b + a.$

$ab = a \times b\,;\; ab + c = (ab) + c\,;\; abc = (ab)\, c.$

§ 5. Powers

Definitions

1. $a \,\varepsilon\, \mathrm{N} \,. \supset .\, a^1 = a.$
2. $a, b \,\varepsilon\, \mathrm{N} \,. \supset .\, a^{b+1} = a^b\, a.$

§ 6. Division

Explanations

The sign / is read *divided by*.
The sign D is read *divides*, or *is a divisor of*.
The sign ᗡ is read *is a multiple of*.
The sign Np is read *prime number*.
The sign π is read *is prime to*.

Definitions

1. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\, b/a = \mathrm{N}\,[x\,\varepsilon]\,(xa = b)$.
2. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!:\, a \,\mathrm{D}\, b \,.\!=\!.\, b/a \,-\!\!= \Lambda$.
3. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!:\, b \,\text{ᗡ}\, a \,.\!=\!.\, a \,\mathrm{D}\, b$.
4. $\mathrm{Np} = \mathrm{N}\,[x\,\varepsilon]\,(\ni \mathrm{D}\, x \,.\! \ni\, > 1 \,.\, \varepsilon < x := \Lambda)$.
5. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!::\, a \,\pi\, b \,.\!\because\!.\!=\!.\!\because\!.\, \ni \mathrm{D}\, a \,.\! \ni \mathrm{D}\, b \,.\! \ni\, > 1 := \Lambda$.
6. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\!\because\!.\, \ni \mathrm{D}\,(a, b) :=: \ni \mathrm{D}\, a \,.\!\cap\!.\, \ni \mathrm{D}\, b$.
7. $a, b \,\varepsilon\, \mathrm{N} \,.\!\supset\!.\!\because\!.\, \ni \text{ᗡ}\,(a, b) :=: \ni \text{ᗡ}\, a \,.\!\cap\!.\, \ni \text{ᗡ}\, b$.

$ab/c = (ab)/c$; $a/b/c = (a/b)/c$; $a/b \times c = (a/b)\,c$.

Letter to Keferstein

RICHARD DEDEKIND

(*1890a*)

Hans Keferstein, *Oberlehrer* in Hamburg, published a paper (*1890*) on the notion of number in which he commented on Frege's (*1884*) and Dedekind's (*1888*) books on the subject. His comments on Dedekind's work, although not entirely negative, included a number of suggestions for amending the text that revealed his lack of real understanding of some fundamental points, for example, the equivalence of two sets. Dedekind felt obliged to answer with an essay (*1890*) in which he showed how pointless the "corrections" were. He sent it to Keferstein on 9 February 1890, with a letter in which he suggested that the Hamburg Mathematical Society, in whose yearly *Mitteilungen* Keferstein's paper had appeared, publish either the essay or, should Keferstein realize that his suggestions were based upon misunderstandings, a declaration to that effect.

Dedekind's essay dealt with three points. The first was an objection of Keferstein's to Dedekind's proof that there exists an infinite set. This proof has often been criticized (see, for instance, below, p. 131); but Keferstein's objection rested upon a wrong argument, a plain confusion of the equivalence relation between sets with their identity, and Dedekind had no difficulty in answering him. The second point was Keferstein's claim that he had found two conflicting definitions of infinite sets in the book; Dedekind pointed out that one was in fact merely a stylistic variant of the other. The third point was the substitution by Keferstein of a new definition of simply infinite sets for that given by Dedekind (*1888*, art. 71). Keferstein's purpose was to avoid the notion of chain, and his proposal amounted in effect to the abandonment of mathematical induction; Dedekind showed that this proposal would bar the possibility of providing an adequate foundation for the theory of natural numbers.

On 14 February 1890 Keferstein acknowledged receipt of the essay, announcing that at the next meeting of the Society he would propose its publication, that he was confident that the proposition would be accepted, that, moreover, he did not consider his criticisms, especially the third one, as mere misunderstandings on his part and would return to them in case the essay should be published.

On 27 February 1890 Dedekind sent to Keferstein a long letter that is a brilliant presentation of the development of his ideas on the notion of natural number. In it he tried to show that his assumptions had not been haphazardly chosen and that each one of them had a profound justification. This is especially true, Dedekind insisted, of the notion of chain, which Keferstein wanted to eliminate. Professor Hao Wang published (*1957*) an English translation of a major part of the letter, with commentaries. The text below is a translation of the whole letter.

On 19 March 1890 Keferstein thanked

Dedekind for the letter and asked his permission to use it in a lecture before the Hamburg Mathematical Society. Dedekind granted this permission in his next letter, dated 1 April 1890, adding a few lines of explanation on the notion of chain.

On 17 November 1890, as publication of the yearly volume of the *Mitteilungen* of the Society was drawing near, Keferstein wrote an "Erwiderung" (*1890b*) that was to follow Dedekind's essay (*1890*) in the volume. But on 19 December 1890 he had to inform Dedekind that the editorial board of the Society had declined to publish Dedekind's essay, as well as Keferstein's rejoinder, the reason invoked being the lack of space and the fact that Dedekind's reply was longer than Keferstein's original criticism of Dedekind's book. Keferstein also announced his intention of publishing in the Society's coming yearly volume a note withdrawing his proposed "corrections" to Dedekind's work. The note appeared in volume 3 of the *Mitteilungen* (p. 31), published in February 1891; it consisted of a few lines incorporated in a report of the 11 October 1890 meeting of the Society.

On 23 December 1890 Dedekind wrote his last letter to Keferstein, acknowledging receipt of his returned manuscript as well as of a copy of Keferstein's "Erwiderung". He expressed his regrets that, although the polemic and the correspondence had taken so much of his time, Keferstein's reply still contained many misunderstandings.

Dedekind's time and efforts were, however, not wasted at all. The controversy produced the letter below, which remains a masterly presentation of his ideas.

Dedekind's essay and the Dedekind–Keferstein correspondence are preserved in the Niedersächsische Staats- und Universitätsbibliothek in Göttingen. They come from the Dedekind estate. Keferstein's letters are the originals, as received by Dedekind; Dedekind's letters and his essay, as well as Keferstein's "Erwiderung", are clean copies in Dedekind's hand. Dedekind's letter of 27 February 1890 is reproduced below with the kind permission of the Library (where it has the classmark: Göttingen, UB, Cod. Ms. Nachlass Dedekind, 13).

Stefan Bauer-Mengelberg translated the parts of the letter omitted from Professor Wang's paper and introduced some changes into the text of Professor Wang's translation. Permission to make use of that translation was granted by Professor Wang and *The journal of symbolic logic*.

My dear Doctor,

I should like to express my sincerest thanks for your kind letter of the 14th of this month and for your willingness to publish my reply. But I would ask you not to rush anything in this matter and to come to a decision only after you have once more carefully read and thoroughly considered the most important definitions and proofs in my essay on numbers, if you have the time. For I think that most probably you will then be converted on all points to my conception and to my treatment of the subject; and this is just what I should value most, since I am convinced that you really have a deep interest in the matter.

In order to further this rapprochement wherever possible, I should like to ask you to lend your attention to the following train of thought, which constitutes the genesis of my essay. How did my essay come to be written? Certainly not in one day; rather, it is a synthesis constructed after protracted labor, based upon a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration. What are the mutually independent fundamental properties of the sequence N, that is, those properties that are not derivable from one another but from which all others follow? And how should we divest these properties of their specifically arithmetic character so that they are subsumed under more general notions and under activities of the understanding *without* which no thinking is possible at all but *with* which a foundation is provided for the reliability and completeness of proofs and for the construction of consistent notions and definitions?

When the problem is posed in this way, one is, I believe, forced to accept the following facts:

(1) The number sequence N is a *system* of individuals, or elements, called numbers. This leads to the general consideration of systems as such (§ 1 of my essay).

(2) The elements of the system N stand in a certain relation to one another; a certain order obtains, which consists, to begin with, in the fact that to each definite number n there corresponds a definite number n', the succeeding, or next greater, number. This leads to the consideration of the general notion of a *mapping* φ of a system (§ 2), and since the image $\varphi(n)$ of every number n is again a *number*, n', and therefore $\varphi(N)$ is a part of N, we are here concerned with the mapping φ of a system N *into itself*, of which we must therefore make a general investigation (§ 4).

(3) Distinct numbers a and b are succeeded by distinct numbers a' and b'; the mapping φ, therefore, has the property of distinctness, or *similarity*[1] (§ 3).

(4) Not every number is a successor n'; in other words, $\varphi(N)$ is a proper part of N. This (together with the preceding) is what makes the number sequence N infinite (§ 5).

(5) And, in particular, the number 1 is the *only* number that does not lie in $\varphi(N)$. Thus we have listed the facts that you (p. 124, ll. 8–14) regard as the complete characterization of an ordered, simply infinite system N.

(6) I have shown in my reply (III),[2] however, that these facts are still far from being adequate for completely characterizing the nature of the number sequence N. All these facts would hold also for every system S that, besides the number sequence N, contained a system T, of arbitrary additional elements t, to which the mapping φ could always be extended while remaining similar and satisfying $\varphi(T) = T$. But such a system S is obviously something quite different from our number sequence N, and I could so choose it that scarcely a single theorem of arithmetic would be preserved in it. What, then, must we add to the facts above in order to cleanse our system S again of such alien intruders t as disturb every vestige of order and to restrict it to N? This was one of the most difficult points of my analysis and its mastery required lengthy reflection. If one presupposes knowledge of the sequence N of natural numbers and, accordingly, allows himself the use of the language of arithmetic, then, of course, he has an easy time of it. He need only say: an element n belongs to the sequence N if and only if, starting with the element 1 and counting on and on steadfastly, that is, going through a finite number of iterations of the mapping φ (see the end of article 131 in my essay), I actually reach the element n at some time; by this procedure, however, I shall never reach an element t outside of the sequence N. But this way of characterizing the distinction between those elements t that are to be ejected from S and those elements n that alone are to remain is surely quite useless for our purpose; it would, after all, contain the most pernicious and obvious kind of vicious

[1] [[See footnote 3, p. 93 above.]]

[2] [[This refers to sec. III in *Dedekind 1890*; see introductory note.]]

circle. The mere words "finally get there at some time", of course, will not do either; they would be of no more use than, say, the words "karam sipo tatura", which I invent at this instant without giving them any clearly defined meaning. Thus, how can I, without presupposing any arithmetic knowledge, give an unambiguous conceptual foundation to the distinction between the elements n and the elements t? Merely through consideration of the *chains* (articles 37 and 44 of my essay), and yet, by means of these, completely! If I wanted to avoid my technical expression "chain" I would say: an element n of S belongs to the sequence N if and only if n is an element of *every* part K of S that possesses the following two properties: (i) the element 1 belongs to K and (ii) the image $\varphi(K)$ is a part of K. In my technical language: N is the intersection [[Gemeinheit]] 1_0, or $\varphi_0(1)$, of all those chains K (in S) to which the element 1 belongs. Only now is the sequence N characterized completely. In passing I would like to make the following remark on this point. Frege's *Begriffsschrift* and *Grundlagen der Arithmetik* came into my possession for the first time for a brief period last summer (1889), and I noted with pleasure that his way of defining the nonimmediate succession of an element upon another in a sequence agrees in *essence* with my notion of chain (articles 37 and 44); only, one must not be put off by his somewhat inconvenient terminology.

(7) After the essential nature of the simply infinite system, whose abstract type is the number sequence N, had been recognized in my analysis (articles 71 and 73), the question arose: does such a system *exist* at all in the realm of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs (articles 66 and 72 of my essay).

(8) After this, too, had been settled, there was the question: does what has been said so far also contain a *method of proof* sufficient to establish, in full generality, propositions that are supposed to hold for *all* numbers n? Yes! The famous method of proof by induction rests upon the secure foundation of the notion of chain (articles 59, 60, and 80 of my essay).

(9) Finally, is it possible also to set up the *definitions* of numerical operations consistently for *all* numbers n? Yes! This is in fact accomplished by the theorem of article 126 of my essay.

Thus the analysis was completed and the synthesis could begin; but this still caused me trouble enough! Indeed the reader of my essay does not have an easy task either; apart from sound common sense, it requires very strong determination to work everything through completely.

I shall now turn to some parts of your paper that I did not mention in my recent reply [[*1890*]] because they are not as important; but perhaps my remarks about them will contribute something more to the clarification of the issue.

(*a*) P. 121, l. 19.[3] Why the mention of a *part* here? I later (article 161 of my essay) ascribe a *number* [[*Anzahl*]] to each *finite* system and to no other.

(*b*) P. 122, l. 8.[4] Here we have a confusion between *mapping* and *map*; instead of

[3] [[Keferstein had written: "In fact he [[Dedekind]] later ascribes each number to a certain part [[Teil]] of such a system...".]]

[4] [[Keferstein had written: "... to the mapping φ of S we can match an inverse mapping $\bar{\varphi}(S')$ ...".]]

"mapping $\bar{\varphi}(S')$" it should be "mapping $\bar{\varphi}$ of the system S'". Not $\bar{\varphi}(S')$ but $\bar{\varphi}$ is a *mapping* (the mapping cartographer) ⟦*Abbildung* (der abbildende Maler)⟧, which generates the *map* $\bar{\varphi}(S') = S$ from the *system* S' (the original). Such confusions can become quite dangerous in our investigations.

(*c*) P. 123, ll. 1–2.[5] These words might perhaps apply to Frege, but they certainly do not apply to me. I define the *number* ⟦*Zahl*⟧ 1 as the basic element of the number sequence without any ambiguity in articles 71 and 73, and, just as unambiguously, I arrive at the *number* ⟦*Anzahl*⟧ 1 in the theorem of article 164 as a consequence of the general definition in article 161. Nothing further *may* be added to this at all if the matter is not to be muddled.

(*d*) P. 123, ll. 29–31.[6] The preceding remark, (*c*), has already taken care of this. And how would the greater reliability and the lesser prolixity shape up in *actual fact*?

(*e*) P. 124, ll. 21–24.[7] The meaning of these lines (as well as of the preceding and subsequent ones) is not quite clear to me. Do they perhaps express the desire that my definition of the number sequence N and of the way in which the element n' follows the element n be propped up, if possible, by an *intuitive* sequence? If so, I would resist that with the utmost determination, since the danger would immediately arise that from such an intuition we might perhaps unconsciously also take as self-evident theorems that must rather be derived quite abstractly from the logical definition of N. If I *call* (article 73) n' the element *following n*, that is only a new *technical expression* by means of which I merely bring some variety into my *language*; this language would sound even more monotonous and repelling if I had to deny myself this variety and were always to call n' only the *map* $\varphi(n)$ of n. But one expression is to *mean* exactly the same as the other.

(*f*) P. 124, l. 33—p. 125, l. 7.[8] The word "merely" ⟦"lediglich"⟧, taken from the third line of my definition in article 73, is obviously meant to indicate the sole *restriction* to which the word "entirely" ⟦"gänzlich"⟧, which occurs just before, is subject;[9]

[5] ⟦Keferstein had written: "In our opinion, both Frege (*1884*, pp. 89–90) and Dedekind, who incidentally derives the notion of cardinal number only from the previously defined notion of ordinal number (*1888*, pp. 21 ⟦article 73⟧ and 54 ⟦article 161⟧), have, when all is said and done, introduced the notion of the number 1 without an adequate definition".⟧

[6] ⟦Keferstein had written: ". . . especially since, by the previous introduction of the number 1, the latter ⟦Dedekind⟧ seems not only to gain in reliability but also to lose in prolixity".⟧

[7] ⟦Keferstein had written: "Since Dedekind does not emphasize this fact ⟦that N can be regarded as a sequence in which $\varphi(n) = n'$ immediately follows n⟧, the notions of sequence and of succession in a sequence turn up in an *apparently* abrupt way in the definition of ordinal numbers that comes at that point".⟧

[8] ⟦Keferstein had written: "When the above comments are properly taken into account, there remains in these propositions at most one point that could give offense, namely, the demand that we *entirely* disregard the particular character of the elements and retain merely their distinguishability, since objects remain distinguishable, after all, only if they still exhibit differences. If we strike out the words 'ihre Unterscheidbarkeit festhält und nur' ⟦see footnote 9⟧, however, the difficulty vanishes, since the relations in which the elements are put with one another by the ordering mapping φ are conceived by precisely a pure mental activity that remains completely independent of the particular character of the objects toward which it is directed".⟧

[9] ⟦The German text to which Dedekind refers reads: "Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung φ geordneten Systems N von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt, in die sie durch die ordnende Abbildung φ zueinander gesetzt sind, so heißen diese Elemente *natürliche Zahlen* oder *Ordinalzahlen* oder auch schlechthin *Zahlen*, und das Grundelement 1 heißt die *Grundzahl* der *Zahlenreihe N*". (*1888*, art. 73).⟧

if one were to remove this restriction—if, in other words, the word "entirely" were to assume its full meaning—then we would lose the distinguishability of the elements, which, after all, is indispensable for the notion of the simply infinite system. This "merely", therefore, does not seem at all superfluous to me, but necessary. I do not understand how it could arouse any objection.

Repeating the wish I expressed at the beginning and begging you to excuse the thoroughness of my discussion, I remain with kindest regards

Yours very truly,

R. Dedekind

27 February 1890
Petrithorpromenade 24