Full name:

1	2	3	4	5	6

4th exam NMAI059 Probability and Statistics 1 – July 21, 2021

Write the number of the problem and your surname on each paper.

On this paper you may also write a pseudonym. In such case, your result will be annonced with this pseudonym (otherwise, with your initials).

Do not write more than one problem on the same sheet of paper!

You have **150 minutes.**

No calculators, cell phones, ... are allowed during the exam. (Please silence your cell phones in advance.)

If the result contains expressions that are difficult to evaluate without a calculator, don't evaluate them (137×173 is as good, if not better, than 23701, you may leave $\Phi^{-1}(0.975)$ unevaluated as well).

Explain in detail all calculations.

You may use one (handwritten) A4 cheat sheet.

After the exam is marked everyone will be offered a grade of 1, ..., 5. You may improve this by one grade in an oral part, that is a 4 can be improved to a 3, a 3 to a 2, a 2 to a 1, but a 5 is a fail for this term of the exam.

Students writing the exam via Zoom must also attend the oral part (presumably also via Zoom) even if they don't desire improving the grade.

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1. (10 points)

In the table is the joint pmf of random variables X, Y. These random variables only take the values indicated in the table.

$x \backslash y$	0	1	2
1	a	1/8	b
0	1/8	3/8	c

(a) Decide if it is possible to fill in the table (i.e., choose a, b, c) so that X and Y are independent.

(b) Fill in the table so that $\mathbb{E}(X)$ is as large as possible.

(c) Fill in the table so that $\mathbb{E}(Y)$ is as large as possible.

(d) Fill in the table so that $\mathrm{var}(X)$ is as small as possible.

2. (10 points) We toss a fair coin repeatedly. Let X_k be the order of the toss in which we first get k heads in a row. That is, for a sequence of tosses HTHHH we have $X_1 = 1$, $X_2 = 4$ a $X_3 = 5$.

(a) Determine $\mathbb{E}(X_1)$.

(b) Determine $\mathbb{E}(X_2)$. To do this, determine first $\mathbb{E}(X_2 \mid \text{first toss was H})$ using $\mathbb{E}(X_2)$. And also $\mathbb{E}(X_2 \mid \text{first two tosses were HT})$ using $\mathbb{E}(X_2)$.

(c) Determine $\mathbb{E}(X_3)$.

3. (10 points) Let X be a random variable with a pdf $f_X(t) = 1/t^2$ for $t \geq 1$ and $f_X(t) = 0$ otherwise.

(a) Verify that it is a pdf.

(b) Determine $\mathbb{E}(X)$.

(c) Compute the cdf, F_X.

(d) Let $Y = 1/X$. What is the cdf of Y?

(e) Determine the pdf of Y. Name its distribution.

4. (10 points) (a) Define the notion cumulative distribution function of a random variable. Define what it means that an event occurs almost surely.

Decide whether one of the following implications is true for all random variables X, Y.

1. $X \leq Y$ a.s. $\Rightarrow F_X(t) \leq F_Y(t)$
2. $X \leq Y$ a.s $\Rightarrow F_Y(t) \leq F_X(t)$
3. $F_X(t) \geq F_Y(t) \Rightarrow X \leq Y$ a.s.

(b) Define the notion variance of a random variable. When is the variance equal to zero?

5. (10 points) Describe methods how to generate a random variable with a given cdf. In particular the basic method (inverse transformation) and possibly also rejection sampling.

6. (10 points) State and prove the theorem about expectation of a product of independent random variables. Variant for discrete random variables is sufficient.