Analytic combinatorics
Lecture 12

June 2, 2021

Series composition inverse

Recall: If $A(x) \in \mathbb{C}[[x]]$ is a power series with $[x^0]A(x) = 0$ and $[x^1]A(x) \neq 0$, then there is a (unique) composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ satisfying $A(B(x)) = B(A(x)) = x$.

Series composition inverse

Recall: If $A(x) \in \mathbb{C}[[x]]$ is a power series with $[x^0]A(x) = 0$ and $[x^1]A(x) \neq 0$, then there is a (unique) composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ satisfying $A(B(x)) = B(A(x)) = x$.

Goal: Compute the coefficients of $B(x)$ from the coefficients of $A(x)$.

Series composition inverse

Recall: If $A(x) \in \mathbb{C}[[x]]$ is a power series with $[x^0]A(x) = 0$ and $[x^1]A(x) \neq 0$, then there is a (unique) composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ satisfying $A(B(x)) = B(A(x)) = x$.

Goal: Compute the coefficients of $B(x)$ from the coefficients of $A(x)$.

Rephrasing the goal: We know that $[x^0]A(x) = 0$, hence $A(x) = xC(x)$ for a series $C(x)$. Moreover, $[x^0]C(x) = [x^1]A(x) \neq 0$, hence $C(x)$ has a multiplicative inverse $F(x) = \frac{1}{C(x)}$. Hence:

$$\begin{aligned} & A(B(x)) = x \\ \iff \quad & B(x)C(B(x)) = x \end{aligned}$$

Series composition inverse

Recall: If $A(x) \in \mathbb{C}[[x]]$ is a power series with $[x^0]A(x) = 0$ and $[x^1]A(x) \neq 0$, then there is a (unique) composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ satisfying $A(B(x)) = B(A(x)) = x$.

Goal: Compute the coefficients of $B(x)$ from the coefficients of $A(x)$.

Rephrasing the goal: We know that $[x^0]A(x) = 0$, hence $A(x) = xC(x)$ for a series $C(x)$. Moreover, $[x^0]C(x) = [x^1]A(x) \neq 0$, hence $C(x)$ has a multiplicative inverse $F(x) = \frac{1}{C(x)}$. Hence:

$$\begin{aligned} & & A(B(x)) &= x \\ \iff & & B(x)C(B(x)) &= x \\ \iff & & B(x) &= \frac{x}{C(B(x))} \end{aligned}$$

Series composition inverse

Recall: If $A(x) \in \mathbb{C}[[x]]$ is a power series with $[x^0]A(x) = 0$ and $[x^1]A(x) \neq 0$, then there is a (unique) composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ satisfying $A(B(x)) = B(A(x)) = x$.

Goal: Compute the coefficients of $B(x)$ from the coefficients of $A(x)$.

Rephrasing the goal: We know that $[x^0]A(x) = 0$, hence $A(x) = xC(x)$ for a series $C(x)$. Moreover, $[x^0]C(x) = [x^1]A(x) \neq 0$, hence $C(x)$ has a multiplicative inverse $F(x) = \frac{1}{C(x)}$. Hence:

$$\begin{aligned} & & A(B(x)) &= x \\ \iff & & B(x)C(B(x)) &= x \\ \iff & & B(x) &= \frac{x}{C(B(x))} \\ \iff & & B(x) &= xF(B(x)) \end{aligned}$$

Series composition inverse

Recall: If $A(x) \in \mathbb{C}[[x]]$ is a power series with $[x^0]A(x) = 0$ and $[x^1]A(x) \neq 0$, then there is a (unique) composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ satisfying $A(B(x)) = B(A(x)) = x$.

Goal: Compute the coefficients of $B(x)$ from the coefficients of $A(x)$.

Rephrasing the goal: We know that $[x^0]A(x) = 0$, hence $A(x) = xC(x)$ for a series $C(x)$. Moreover, $[x^0]C(x) = [x^1]A(x) \neq 0$, hence $C(x)$ has a multiplicative inverse $F(x) = \frac{1}{C(x)}$. Hence:

$$\begin{aligned} & & A(B(x)) &= x \\ \iff & & B(x)C(B(x)) &= x \\ \iff & & B(x) &= \frac{x}{C(B(x))} \\ \iff & & B(x) &= xF(B(x)) \end{aligned}$$

New goal: For a power series $F(x) \in \mathbb{C}[[x]]$ with $[x^0]F(x) \neq 0$, find the (unique) power series $B(x)$ satisfying $B(x) = xF(B(x))$.

Series composition inverse

Recall: If $A(x) \in \mathbb{C}[[x]]$ is a power series with $[x^0]A(x) = 0$ and $[x^1]A(x) \neq 0$, then there is a (unique) composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ satisfying $A(B(x)) = B(A(x)) = x$.

Goal: Compute the coefficients of $B(x)$ from the coefficients of $A(x)$.

Rephrasing the goal: We know that $[x^0]A(x) = 0$, hence $A(x) = xC(x)$ for a series $C(x)$. Moreover, $[x^0]C(x) = [x^1]A(x) \neq 0$, hence $C(x)$ has a multiplicative inverse $F(x) = \frac{1}{C(x)}$. Hence:

$$\begin{aligned} & & A(B(x)) &= x \\ \iff & & B(x)C(B(x)) &= x \\ \iff & & B(x) &= \frac{x}{C(B(x))} \\ \iff & & B(x) &= xF(B(x)) \end{aligned}$$

New goal: For a power series $F(x) \in \mathbb{C}[[x]]$ with $[x^0]F(x) \neq 0$, find the (unique) power series $B(x)$ satisfying $B(x) = xF(B(x))$.

Remark: If $[x^0]F(x) = 0$ then the equation $B(x) = xF(B(x))$ has the (trivial) unique solution $B(x) = 0$.

Lagrange inversion formula(s)

Theorem (Lagrange inversion formula)

Suppose $F(x)$ is a power series with $[x^0]F(x) \neq 0$. Let $B(x) \in \mathbb{C}[[x]]$ be the solution of the functional equation $B(x) = xF(B(x))$. Then the following holds:

1. *For any $n \in \mathbb{N}$,*
$$[x^n]B(x) = \frac{1}{n}[x^{n-1}]F(x)^n.$$
2. *For any $k, n \in \mathbb{N}$,*
$$[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n.$$
3. *For any $G(x) \in \mathbb{C}[[x]]$ and $n \in \mathbb{N}$,*
$$[x^n]G(B(x)) = \frac{1}{n}[x^{n-1}]\left(F(x)^n \frac{d}{dx}G(x)\right).$$

Lagrange inversion formula(s)

Theorem (Lagrange inversion formula)

Suppose $F(x)$ is a power series with $[x^0]F(x) \neq 0$. Let $B(x) \in \mathbb{C}[[x]]$ be the solution of the functional equation $B(x) = xF(B(x))$. Then the following holds:

1. *For any $n \in \mathbb{N}$,*
$$[x^n]B(x) = \frac{1}{n}[x^{n-1}]F(x)^n.$$
2. *For any $k, n \in \mathbb{N}$,*
$$[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n.$$
3. *For any $G(x) \in \mathbb{C}[[x]]$ and $n \in \mathbb{N}$,*
$$[x^n]G(B(x)) = \frac{1}{n}[x^{n-1}]\left(F(x)^n \frac{d}{dx}G(x)\right).$$

Note: $2 \Rightarrow 1$ by taking $k = 1$, and $3 \Rightarrow 2$ by taking $G(x) = x^k$.

$$\frac{d}{dx}x^k = kx^{k-1}$$

$$[x^{n-1}]F^n(x)\cdot kx^{k-1} = k[x^{n-k}]F^n(x)$$

Lagrange inversion formula(s)

Theorem (Lagrange inversion formula)

Suppose $F(x)$ is a power series with $[x^0]F(x) \neq 0$. Let $B(x) \in \mathbb{C}[[x]]$ be the solution of the functional equation $B(x) = xF(B(x))$. Then the following holds:

1. *For any $n \in \mathbb{N}$,*
$$[x^n]B(x) = \frac{1}{n}[x^{n-1}]F(x)^n.$$
2. *For any $k, n \in \mathbb{N}$,*
$$[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n.$$
3. *For any $G(x) \in \mathbb{C}[[x]]$ and $n \in \mathbb{N}$,*
$$[x^n]G(B(x)) = \frac{1}{n}[x^{n-1}]\left(F(x)^n \frac{d}{dx} G(x)\right).$$

Note: $2 \Rightarrow 1$ by taking $k = 1$, and $3 \Rightarrow 2$ by taking $G(x) = x^k$.

Note: $2 \Rightarrow 3$ by linearity: for $G(x) = \sum_{k=0}^{\infty} g_k x^k$, we get

$$\underline{[x^n]G(B(x))} = [x^n]\underline{\sum_{k=0}^{\infty} g_k B(x)^k} = [x^n]\underline{\sum_{k=1}^{\infty} g_k B(x)^k} = \underline{\sum_{k=1}^{\infty} g_k \frac{k}{n}[x^{n-k}]F(x)^n} =$$
$$\sum_{k=1}^{\infty} g_k \frac{k}{n}[x^{n-1}]x^{k-1}F(x)^n = \frac{1}{n}[x^{n-1}]F(x)^n \sum_{k=1}^{\infty} k g_k x^{k-1} = \underline{\frac{1}{n}[x^{n-1}]\left(F(x)^n \frac{d}{dx} G(x)\right)}.$$

Computations with residues

Recall: If f is a complex function meromorphic in 0, then there is a $d \in \mathbb{Z}$ such that on a punctured neighborhood of 0, f is equal to a Laurent series $f(z) = \sum_{n \geq d} f_n z^n$. The coefficient f_{-1} in this series is the residue of f in 0, denoted $\mathrm{Res}_0(f)$.

$d = -3$

$$f(z) = \frac{f_{-3}}{z^3} + \frac{f_{-2}}{z^2} + \frac{f_{-1}}{z} + f_0 + f_1 z + \dots$$

Computations with residues

Recall: If f is a complex function meromorphic in 0, then there is a $d \in \mathbb{Z}$ such that on a punctured neighborhood of 0, f is equal to a Laurent series $f(z) = \sum_{n \geq d} f_n z^n$. The coefficient f_{-1} in this series is the residue of f in 0, denoted $\mathrm{Res}_0(f)$.

Lemma (Derivatives have no residues)

With f as above, if f has a primitive function on a punctured neighborhood of 0, then $\mathrm{Res}_0(f) = 0$.

Computations with residues

Recall: If f is a complex function meromorphic in 0, then there is a $d \in \mathbb{Z}$ such that on a punctured neighborhood of 0, f is equal to a Laurent series $f(z) = \sum_{n \geq d} f_n z^n$. The coefficient f_{-1} in this series is the residue of f in 0, denoted $\mathrm{Res}_0(f)$.

Lemma (Derivatives have no residues)

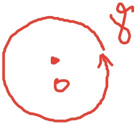
With f as above, if f has a primitive function on a punctured neighborhood of 0, then $\mathrm{Res}_0(f) = 0$.

Proof.

For a circle γ around 0 of small enough radius, we have

$$\mathrm{Res}_0(f) = \frac{1}{2\pi i} \int_\gamma f = 0,$$

where the intergal is 0, because f has a primitive function. $\square$



$$\left(\sum_{n=d}^{\infty} f_n z^n \right)' = \sum_{n=d}^{\infty} n f_n z^{n-1}$$

res = 0

Substitution for residues

Lemma (Substitution rule for residues)

With f as above, if $g(z)$ is analytic on 0 with $g(0) = 0$ and $g'(0) \neq 0$, then $\mathrm{Res}_0(f(z)) = \mathrm{Res}_0(f(g(z))g'(z))$.

$\| \qquad z \to g(z) \qquad \|$

$$\frac{1}{2\pi i}\int_{\gamma} f = \frac{1}{2\pi i}\int_{g^{-1}(\gamma)} f(g(z)) \cdot g'(z)$$

Substitution for residues

Lemma (Substitution rule for residues)

With f as above, if $g(z)$ is analytic on 0 with $g(0) = 0$ and $g'(0) \neq 0$, then $\mathrm{Res}_0(f(z)) = \mathrm{Res}_0(f(g(z))g'(z))$.

Proof.

With $f(z) = \sum_{n \geq d} f_n z^n$ and $g(z) = \sum_{n \geq 1} g_n z^n$, we get

$$\mathrm{Res}_0(f(g(z))g'(z)) = \mathrm{Res}_0\left(\sum_{n \geq d} f_n g(z)^n g'(z)\right)$$

Lemma (Substitution rule for residues)

With f as above, if $g(z)$ is analytic on 0 with $g(0) = 0$ and $g'(0) \neq 0$, then $\mathrm{Res}_0(f(z)) = \mathrm{Res}_0(f(g(z))g'(z))$.

Proof.

With $f(z) = \sum_{n \geq d} f_n z^n$ and $g(z) = \sum_{n \geq 1} g_n z^n$, we get

$$\mathrm{Res}_0(f(g(z))g'(z)) = \mathrm{Res}_0\left(\sum_{n \geq d} \underbrace{f_n g(z)^n g'(z)}\right)$$

$$= \underbrace{\mathrm{Res}_0\left(\sum_{\substack{n \geq d \\ n \neq -1}} \overbrace{\left(\frac{f_n}{n+1} g(z)^{n+1}\right)'}\right)}_{0} + \mathrm{Res}_0\left(f_{-1} \frac{g'(z)}{g(z)}\right)$$

Substitution for residues

Lemma (Substitution rule for residues)

With f as above, if $g(z)$ is analytic on 0 with $g(0) = 0$ and $g'(0) \neq 0$, then $\mathrm{Res}_0(f(z)) = \mathrm{Res}_0(f(g(z))g'(z))$.

Proof.

With $f(z) = \sum_{n \geq d} f_n z^n$ and $g(z) = \sum_{n \geq 1} g_n z^n$, we get

$$\begin{aligned}
\mathrm{Res}_0(f(g(z))g'(z)) &= \mathrm{Res}_0\left(\sum_{n \geq d} f_n g(z)^n g'(z)\right) \\
&= \mathrm{Res}_0\left(\sum_{\substack{n \geq d \\ n \neq -1}} \left(\frac{f_n}{n+1} g(z)^{n+1}\right)'\right) + \mathrm{Res}_0\left(f_{-1}\frac{g'(z)}{g(z)}\right) \\
&= \mathrm{Res}_0\left(f_{-1}\frac{g_1 + 2g_2 z + 3g_3 z^2 + \cdots}{g_1 z + g_2 z^2 + g_3 z^3 + \cdots}\right)
\end{aligned}$$

Substitution for residues

Lemma (Substitution rule for residues)

With f as above, if $g(z)$ is analytic on 0 with $g(0) = 0$ and $g'(0) \neq 0$, then $\mathrm{Res}_0(f(z)) = \mathrm{Res}_0(f(g(z))g'(z))$.

Proof.

With $f(z) = \sum_{n\geq d} f_n z^n$ and $g(z) = \sum_{n\geq 1} g_n z^n$, we get

$$
\begin{aligned}
\mathrm{Res}_0(f(g(z))g'(z)) &= \mathrm{Res}_0\left(\sum_{n\geq d} f_n g(z)^n g'(z)\right)\\
&= \mathrm{Res}_0\left(\sum_{\substack{n\geq d\\ n\neq -1}}\left(\frac{f_n}{n+1}g(z)^{n+1}\right)'\right) + \mathrm{Res}_0\left(f_{-1}\frac{g'(z)}{g(z)}\right)\\
&= \mathrm{Res}_0\left(f_{-1}\frac{g_1 + 2g_2 z + 3g_3 z^2 + \cdots}{g_1 z + g_2 z^2 + g_3 z^3 + \cdots}\right)\\
&= \mathrm{Res}_0\left(\frac{f_{-1}}{z}\cdot \underbrace{\frac{g_1 + 2g_2 z + 3g_3 z^2 + \cdots}{g_1 + g_2 z + g_3 z^2 + \cdots}}_{1 + a_1 z + a_2 z^2 + \ldots}\right)
\end{aligned}
$$

Substitution for residues

Lemma (Substitution rule for residues)

With f as above, if $g(z)$ is analytic on 0 with $g(0) = 0$ and $g'(0) \neq 0$, then $\mathrm{Res}_0(f(z)) = \mathrm{Res}_0(f(g(z))g'(z))$.

Proof.

With $f(z) = \sum_{n \geq d} f_n z^n$ and $g(z) = \sum_{n \geq 1} g_n z^n$, we get

$$
\begin{aligned}
\underline{\mathrm{Res}_0(f(g(z))g'(z))} &= \mathrm{Res}_0\left(\sum_{n \geq d} f_n g(z)^n g'(z)\right) \\
&= \mathrm{Res}_0\left(\sum_{\substack{n \geq d \\ n \neq -1}} \left(\frac{f_n}{n+1} g(z)^{n+1}\right)'\right) + \mathrm{Res}_0\left(f_{-1}\frac{g'(z)}{g(z)}\right) \\
&= \mathrm{Res}_0\left(f_{-1}\frac{g_1 + 2g_2 z + 3g_3 z^2 + \cdots}{g_1 z + g_2 z^2 + g_3 z^3 + \cdots}\right) \\
&= \mathrm{Res}_0\left(\frac{f_{-1}}{z} \cdot \frac{g_1 + 2g_2 z + 3g_3 z^2 + \cdots}{g_1 + g_2 z + g_3 z^2 + \cdots}\right) \\
&= f_{-1} = \underline{\mathrm{Res}_0(f)}.
\end{aligned}
$$

$\square$

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$. $n, k \in \mathbb{N}$

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Since $F(0) \neq 0$, $\frac{x}{F(x)}$ is analytic in 0.

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Since $F(0) \neq 0$, $\frac{x}{F(x)}$ is analytic in 0.

Since $B(x)$ is a composition inverse of $\frac{x}{F(x)}$, it is analytic in 0 as well.

$\square$

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Since $F(0) \neq 0$, $\frac{x}{F(x)}$ is analytic in 0.

Since $B(x)$ is a composition inverse of $\frac{x}{F(x)}$, it is analytic in 0 as well.

Proof of LIF.

$$\frac{k}{n}[x^{n-k}]F(x)^n = \frac{k}{n}\mathrm{Res}_0 \frac{F(x)^n}{x^{n-k+1}} = \frac{1}{n}\mathrm{Res}_0\left(kx^{k-1}\frac{F(x)^n}{x^n}\right)$$

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Since $F(0) \neq 0$, $\frac{x}{F(x)}$ is analytic in 0.

Since $B(x)$ is a composition inverse of $\frac{x}{F(x)}$, it is analytic in 0 as well.

Proof of LIF.

$$\begin{aligned}\frac{k}{n}[x^{n-k}]F(x)^n &= \frac{k}{n}\,\mathrm{Res}_0\,\frac{F(x)^n}{x^{n-k+1}} = \frac{1}{n}\,\mathrm{Res}_0\left(kx^{k-1}\frac{F(x)^n}{x^n}\right)\\ &= \frac{1}{n}\,\mathrm{Res}_0\left(kB(x)^{k-1}\frac{F(B(x))^n}{B(x)^n}B'(x)\right) \qquad \text{(substitute } x \to B(x))\end{aligned}$$

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Since $F(0) \neq 0$, $\frac{x}{F(x)}$ is analytic in 0.

Since $B(x)$ is a composition inverse of $\frac{x}{F(x)}$, it is analytic in 0 as well.

Proof of LIF.

$$\begin{aligned}
\frac{k}{n}[x^{n-k}]F(x)^n &= \frac{k}{n}\operatorname{Res}_0 \frac{F(x)^n}{x^{n-k+1}} = \frac{1}{n}\operatorname{Res}_0\left(kx^{k-1}\frac{F(x)^n}{x^n}\right) \\
&= \frac{1}{n}\operatorname{Res}_0\left(kB(x)^{k-1}\frac{F(B(x))^n}{B(x)^n}B'(x)\right) && \text{(substitute } x \to B(x)) \\
&= \frac{1}{n}\operatorname{Res}_0\left(kB(x)^{k-1}\frac{1}{x^n}B'(x)\right) && \left(\text{use } \frac{B(x)}{F(B(x))} = x\right)
\end{aligned}$$

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Since $F(0) \neq 0$, $\frac{x}{F(x)}$ is analytic in 0.

Since $B(x)$ is a composition inverse of $\frac{x}{F(x)}$, it is analytic in 0 as well.

Proof of LIF.

$$
\begin{aligned}
\frac{k}{n}[x^{n-k}]F(x)^n &= \frac{k}{n}\,\mathrm{Res}_0\,\frac{F(x)^n}{x^{n-k+1}} = \frac{1}{n}\,\mathrm{Res}_0\left(kx^{k-1}\frac{F(x)^n}{x^n}\right) \\
&= \frac{1}{n}\,\mathrm{Res}_0\left(kB(x)^{k-1}\frac{F(B(x))^n}{B(x)^n}B'(x)\right) && \text{(substitute } x \to B(x)\text{)} \\
&= \frac{1}{n}\,\mathrm{Res}_0\left(kB(x)^{k-1}\frac{1}{x^n}B'(x)\right) && \left(\text{use } \frac{B(x)}{F(B(x))} = x\right) \\
&= \frac{1}{n}[x^{n-1}]kB(x)^{k-1}B'(x) = \frac{1}{n}[x^{n-1}](B(x)^k)'
\end{aligned}
$$

Back to Lagrange inversion formula

Recall: LIF says that $[x^n]B(x)^k = \frac{k}{n}[x^{n-k}]F(x)^n$, where $B(x)$ is the solution of $B(x) = xF(B(x))$ and $F(x)$ is a given series with $[x^0]F(x) \neq 0$.

Note: Both $[x^n]B(x)^k$ and $[x^{n-k}]F(x)^n$ only depend on the coefficients of F of degree at most n. Hence we may assume that F is a polynomial, and in particular an analytic function.

Since $F(0) \neq 0$, $\frac{x}{F(x)}$ is analytic in 0.

Since $B(x)$ is a composition inverse of $\frac{x}{F(x)}$, it is analytic in 0 as well.

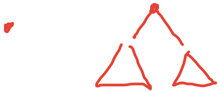
Proof of LIF.

$$\begin{aligned}
\frac{k}{n}[x^{n-k}]F(x)^n &= \frac{k}{n}\operatorname{Res}_0 \frac{F(x)^n}{x^{n-k+1}} = \frac{1}{n}\operatorname{Res}_0\left(kx^{k-1}\frac{F(x)^n}{x^n}\right) \\
&= \frac{1}{n}\operatorname{Res}_0\left(kB(x)^{k-1}\frac{F(B(x))^n}{B(x)^n}B'(x)\right) && \text{(substitute } x \to B(x)\text{)} \\
&= \frac{1}{n}\operatorname{Res}_0\left(kB(x)^{k-1}\frac{1}{x^n}B'(x)\right) && \left(\text{use } \frac{B(x)}{F(B(x))} = x\right) \\
&= \frac{1}{n}[x^{n-1}]kB(x)^{k-1}B'(x) = \frac{1}{n}[x^{n-1}](B(x)^k)' \\
&= [x^n]B(x)^k
\end{aligned}$$

$\square$

Catalan trees revisited

Recall from Lecture 10: A binary tree is either a single leaf node, or an internal root node together with an ordered pair of subtrees, which are both binary trees. Let t_n be the number of binary trees with n internal nodes. Let us deduce a formula for t_n using LIF.



Recall from Lecture 10: A binary tree is either a single leaf node, or an internal root node together with an ordered pair of subtrees, which are both binary trees. Let t_n be the number of binary trees with n internal nodes. Let us deduce a formula for t_n using LIF.

The OGF $T(z) = \sum_{n=0}^{\infty} t_n z^n$ satisfies $T(z) = 1 + zT^2(z)$.

$$\{\text{trees}\} = \{\bullet\} \,\dot\cup\, \{ T_1 \quad T_2 \}$$

$$\{\text{trees}\} = \{\circ\} \,\dot\cup\, \{\text{root}\} \times \{\text{trees}\} \times \{\text{trees}\}$$

$$T(z) = 1 + z \cdot T(z) \cdot T(z)$$

$$T(z) = \frac{1-\sqrt{1-4z}}{2z}$$

Catalan trees revisited

Recall from Lecture 10: A binary tree is either a single leaf node, or an internal root node together with an ordered pair of subtrees, which are both binary trees. Let t_n be the number of binary trees with n internal nodes. Let us deduce a formula for t_n using LIF.

$TB(x) = xF(B(x))$

The OGF $T(z) = \sum_{n=0}^{\infty} t_n z^n$ satisfies $T(z) = 1 + zT^2(z)$.

Define $T^+(z) = T(z) - 1$ to be the OGF of trees with at least one internal node. The above formula gives $T^+(z) = z(T^+(z) + 1)^2$.

Catalan trees revisited

Recall from Lecture 10: A binary tree is either a single leaf node, or an internal root node together with an ordered pair of subtrees, which are both binary trees. Let t_n be the number of binary trees with n internal nodes. Let us deduce a formula for t_n using LIF.

The OGF $T(z) = \sum_{n=0}^{\infty} t_n z^n$ satisfies $T(z) = 1 + zT^2(z)$.

Define $T^+(z) = T(z) - 1$ to be the OGF of trees with at least one internal node. The above formula gives $T^+(z) = z(T^+(z)+1)^2$.

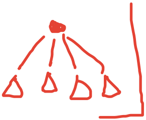
In particular, $T^+(z) = zF(T^+(z))$ for $F(z) = (z+1)^2$. Hence, by LIF,

$n \geq 1$

$$\begin{aligned} t_n = [z^n]T^+(z) &= \frac{1}{n}[z^{n-1}](z+1)^{2n} \\ &= \frac{1}{n}\binom{2n}{n-1} = \underbrace{\frac{1}{n+1}\binom{2n}{n}} = C_n \end{aligned}$$

Plane trees and plane forests

A plane tree consists of a root node together with an ordered d-tuple of subtrees, for some $d \in \mathbb{N}_0$. The size of a plane tree is its number of nodes (in particular, each plane tree has positive size). Let p_n be the number of plane trees of size n.

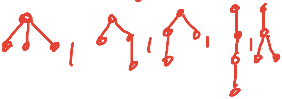


$p_0 = 0$

$p_1 = 1$ 

$p_2 = 1$ 

$p_3 = 2$ 

$p_4 = 5$ 

Plane trees and plane forests

A plane tree consists of a root node together with an ordered d-tuple of subtrees, for some $d \in \mathbb{N}_0$. The size of a plane tree is its number of nodes (in particular, each plane tree has positive size). Let p_n be the number of plane trees of size n.

A plane forest with k components is an ordered k-tuple of plane trees. Let f_n be the number of plane forests with k components that have total size n (k is a fixed constant).

A plane tree consists of a root node together with an ordered d-tuple of subtrees, for some $d \in \mathbb{N}_0$. The size of a plane tree is its number of nodes (in particular, each plane tree has positive size). Let p_n be the number of plane trees of size n.

A plane forest with k components is an ordered k-tuple of plane trees. Let f_n be the number of plane forests with k components that have total size n (k is a fixed constant).

Goal: Find an explicit formula for p_n and f_n. $(k=1)$

$$P(z) = \sum_{n\geq 1} p_n z^n = z + zP(z) + zP^2(z) + \ldots$$

(root, subtree)

$$= z\left(\sum_{k=0}^{\infty} P^k(z)\right) = z \cdot \frac{1}{1-P(z)}$$

$$= zF(P(z)), \quad F(z) = \frac{1}{1-z}$$

$$p_n = [x^n]\, P(x)$$

$$= \frac{1}{n}[x^{n-1}] \frac{1}{(1-x)^n}$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k x^k$$

$$= \frac{1}{n}\binom{-n}{n-1}(-1)^{n-1} = \frac{1}{n} \cdot \frac{(-n)\cdot\ldots\cdot\cancel{(-2n+1)}}{(n-1)!}(-1)^{n-1}$$

$(-2n+2)$

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

$$[x^a]\frac{1}{(1-x)^n} = [x^a](1+x+x^2+\dots)^n = \text{number of}$$

$a \in \mathbb{N}_0$ possibilities to obtain <u>a</u> as a sum of n nonnegative integers $= \binom{a+n-1}{a} = \binom{a+n-1}{n-1}$

$\bullet\,|\,|\,\bullet\;\bullet\,|\,\bullet\;\bullet\,|\,\bullet\,\bullet\,\bullet$

$1+0+2+2+3$

---

k components: f_n, $\sum_{n=0}^{\infty} f_n x^n$ in terms $P(x)$: $P(x)^k$

$\{\text{forests}\} = \{$

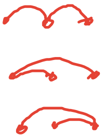
$$f_n = [x^n]\,P(x)^k = \frac{k}{n}[x^{n-k}]\frac{1}{(1-x)^n} = \frac{k}{n}\binom{2n-k-1}{n-k}$$

Labelled trees

A rooted tree of size n is a tree on the vertex set $[n]$ with one vertex designated as root. A rooted forest of size n is a graph on the vertex set $[n]$ whose every component is a rooted tree. Let r_n be the number of rooted trees on the vertices $[n]$, and let g_n be the number of rooted forests with k components on $[n]$ (k again fixed).

$r_0 = 0$, $r_1 = 1$, $r_2 = 2$

$r_3 = 9$



$\times 3$

$\times 3$

$\times 3$

A rooted tree of size n is a tree on the vertex set $[n]$ with one vertex designated as root. A rooted forest of size n is a graph on the vertex set $[n]$ whose every component is a rooted tree. Let r_n be the number of rooted trees on the vertices $[n]$, and let g_n be the number of rooted forests with k components on $[n]$ (k again fixed).

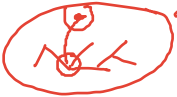
Goal: Find an explicit formula for r_n and g_n.

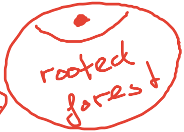
$$R(x) = \sum_{n=0}^{\infty} \frac{r_n}{n!} x^n$$

root has $\deg = 1$: $x \cdot R(x)$

$\deg = 2$: $\frac{1}{2} x \, R^2(x)$

$\deg = k$: $\frac{1}{k!} x \, R^k(x)$





{rooted trees w. root of deg 1} = {root} ⊗ {rooted tree}

{ '' deg 2} = {root} ⊗ {rooted tree} ⊗ {rooted tree} / 2

$$R(x) = \sum_{k=0}^{\infty} \frac{x \cdot R(x)^k}{k!} = x \cdot e^{R(x)}$$

$$R(x) = x\, e^{R(x)} = x\, F(R(x)) \quad \text{with } F = e^x$$

rooted forests with k components have EGF

$$\frac{R(x)^k}{k!}$$

$$r_n = n!\,[x^n]\,R(x) =$$

$$= n!\left(\frac{1}{n}[x^{n-1}]\,(e^x)^n\right) =$$

$$= n!\left(\frac{1}{n}[x^{n-1}]\,e^{nx}\right)$$

$$= n!\left(\frac{1}{n}\,\frac{n^{n-1}}{(n-1)!}\right)$$

$$= n^{n-1}$$ $= n \cdot$ number of unrooted trees

$$= n \cdot n^{n-2}$$