(1) (a)

$$P(\text{správně} \mid \text{umí}) = 1$$

$$P(\text{správně} \mid \text{neumí}) = \frac{1}{3}$$

$$P(\text{umí}) = \frac{1}{2}$$

zadání

$$P(\text{umí} \mid \text{nesprávně}) = 0$$

$$P(\text{umí} \mid \text{správně}) = \frac{P(\text{umí}) . P(\text{spr} \mid \text{umí})}{P(\text{umí})\, P(\text{spr} \mid \text{umí}) + P(\text{neumí}) . P(\text{spr.} \mid \text{neumí})}$$

$$= \frac{\frac{1}{2} \cdot 1}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{3}} = \frac{1/2}{2/3} = \boxed{\frac{3}{4}}$$

(b) Každou otázku umí s pst $\frac{1}{2}$, tj.

$U \sim Bin(10, \frac{1}{2})$ nebo ...

$$P(U = k) = \binom{10}{k} \cdot \frac{1}{2^{10}}$$

$\mathbb{E}(U) = 5$ $\sigma = 1.6$

$var(U) = 10 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2.5$

$\boxed{U \leq 6 \text{ určitě}}$

1 2 3 4 5 6 spr.
7 -- 10 špatně ---> neumí

(c) $U \mid S = 6 \sim Bin(6, \frac{3}{4})$, tj. $\cancel{P(U = k)}$ $P(U = k \mid spr. = 6) = \binom{6}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{6-k}$

$$\mathbb{E}(U \mid S = 6) = 6 \cdot \frac{3}{4} = \frac{9}{2} = 4.5 \checkmark$$

$$var(U \mid S = 6) = 6 \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{18}{16}$$ $\sigma = 1.06$

② $P(\Theta=0.3)=P(\Theta=0.7)=P(\Theta=0.95)=\frac{1}{3}$ apriorní pravd. rozd. = prior

ⓐ $S/\Theta=p = \text{Binom}\left(10, \bar{p}+(1-p)\cdot\frac{1}{3}\right)$

$\boxed{p(\text{spr.})=\Theta+(1-\Theta)\cdot\frac{1}{3}}$

$\bar{p}=p+(1-p)\cdot\frac{1}{3}$ = pravd. spr. odp. $/\Theta=p$

$P(S=k/\Theta=p)=\binom{10}{k}\bar{p}^k(1-\bar{p})^{10-k}$

$P(\Theta=p/S=k)=\dfrac{P(S=k/\Theta=p)\cdot\boxed{P(\Theta=p)}}{\sum\limits_{p'\in\{0.3,0.7,0.95\}}P(S=k/\Theta=p')\cdot P(\Theta=p')}=\dfrac{\bar{p}^k(1-\bar{p})^{10-k}\cancel{\frac{1}{3}}\cancel{\binom{10}{k}}}{\sum\limits_{p'\in\{0.3,0.7,0.95\}}\bar{p}'^k(1-\bar{p}')^{10-k}\cancel{\frac{1}{3}}\cancel{\binom{10}{k}}}$

a posteriori

tři skupiny

MAX. pouze volbou P

MAP pro který $p\in\{0.3, 0.7, 0.95\}$ je $P(\Theta=p/S=k)$ maximální?

početně jednodušší — Σ ve jmenovateli nezávisí na p

... tj. stačí max. $P(S=k/\Theta=p)$

chceme najít $p\in\{0.3, 0.7, 0.95\}$ pro které je $\bar{p}^k(1-\bar{p})^{10-k}$ maximální

ⓑ $P(\text{umí}/\text{správně})=\dfrac{\Theta\cdot 1}{\Theta\cdot 1+(1-\Theta)\cdot\frac{1}{3}}=\dfrac{\Theta}{\frac{1}{3}+\frac{2}{3}\Theta}$ $\quad P(\text{umí}/\text{nespr.})=0$

tři volby

Dáno $k=5$.

$\Theta = \begin{cases}0.3\\0.7\\0.95\end{cases}$

$U/\Theta=p, S=k = \boxed{\text{Bin}\left(k, \dfrac{p}{\frac{1}{3}+\frac{2}{3}p}\right)}$

$\boxed{k=5}$

MAP: chceme l: $P(U=l/S=5)$ bylo max.

$E(U/\Theta=p, S=k)=k\cdot\dfrac{p}{\frac{1}{3}+\frac{2}{3}p}$

$\frac{1}{3}P(U=l/S=5 \,\&\, \Theta=0.3)+$

$+\frac{1}{3}P(\ldots\ldots\ \Theta=0.7)+$

$\frac{1}{3}P(\ldots\ldots\ \Theta=0.95)$

$k=5$

$\binom{k}{l}\left(\dfrac{p}{\frac{1}{3}+\frac{2}{3}p}\right)^l\cdot\left(1-\dfrac{p}{\frac{1}{3}+\frac{2}{3}p}\right)^{k-l}$

podm. pravd.

$E(U/S=5)=\frac{1}{3}E(U/S=5\,\&\,\Theta=0.3)+$

$+\frac{1}{3}E(U/\ldots\ \Theta=0.7)+$

$+\frac{1}{3}\ \ldots\ \ldots\ 0.95$

$=\frac{1}{3}5\cdot\dfrac{0.3}{\frac{1}{3}+\frac{2}{3}0.3}+\frac{1}{3}\cdot 5\cdot\dfrac{0.7}{\ldots}+\frac{1}{3}5\cdot\dfrac{0.95}{\ldots}$

③ a) 1. $\boxed{p}$ ○○● 2. $1-p$ ○●●

Předpokládejme, že jsme vytáhli bílý míček.

$$P(b.m./1.kr.) = \frac{2}{3}, \quad P(b.m./2.kr.) = \frac{1}{3}$$

$$P(1.kr.) = p, \quad P(2.kr.) = 1-p$$

Bayesova věta

$$P(1.kr./b.m.) = \frac{P(bm/1.kr.)\cdot P(1.kr.)}{P(bm/1.kr.)P(1.kr.) + P(bm/2.kr.)P(2.kr.)}$$

$$= \frac{\frac{2}{3}\cdot p}{\frac{2}{3}p + \frac{1}{3}(1-p)} = \frac{2p}{1+p}$$

$$P(2.kr./b.m.) = 1 - \frac{2p}{1+p} = \frac{1-p}{1+p}$$

Pokud (jsme vytáhli b.m.) $\frac{2p}{1+p} > \frac{1}{2}$, řekneme 1. kr., jinak 2. kr.

$$4p > 1+p, \quad \boxed{p > \frac{1}{3}}$$

Pro černý míček obdobně ... s hranicí $p = \frac{2}{3}$. Závěr: $p \le \frac{1}{3}$ -- řekneme 2. kr.

$p \ge \frac{2}{3}$ ---- 1. kr.

$p \in (\frac{1}{3}, \frac{2}{3})$ ..... řekneme 1. kr. při bílém míčku a 2. kr. při černém.

b) $p = \frac{1}{2}$ $P(1.kr./b.m.) = \frac{1}{3/2} = \frac{2}{3}$

$$P(\text{MAP pravděp. odhadu chyba}) = P(2.kr. \,\&\, b.m.) + P(1.kr. \,\&\, č.m.)$$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{3}$$ ← chyba aposteriorního odhadu

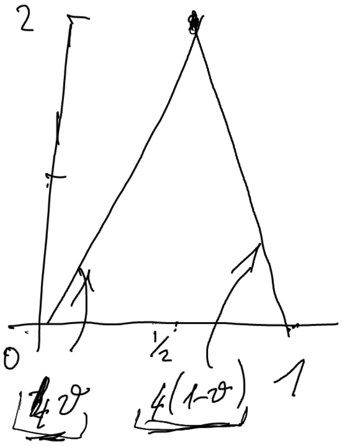
bez tahání míčku: $P(\text{chyba}) = \frac{1}{2}$ ← chyba apriorního odhadu

obecně $p\frac{1}{3} + (1-p)\frac{1}{3} = \frac{1}{3}$ ... P(chyby algo b.m. → 1. kr., č.m. → 2. kr.)

Pokud $p < \frac{1}{3}$, MAP = vždy 2. kr.

$$P(\text{chyba}) = p$$

(4)



$f = f_\Theta$

$$\boxed{X \sim Bin(n, \Theta)}$$

MAX.

$$f_{\Theta|X}(\vartheta|k) = \frac{f_\Theta(\vartheta)\cdot P_{X|\Theta}(k|\vartheta)}{\int_0^1 f_\Theta(\vartheta')\, P_{X|\Theta}(k|\vartheta')\, d\vartheta'}$$

$P(X=k|\Theta=\vartheta)$

$\binom{n}{k}\vartheta^k(1-\vartheta)^{n-k}$

tohle chceme max

nezávisí na ϑ

konst.

$$= \frac{f(\vartheta)\cdot\binom{n}{k}\vartheta^k(1-\vartheta)^{n-k}}{\int_0^1 f(\vartheta')\binom{n}{k}\vartheta'^k(1-\vartheta')^{n-k}\, d\vartheta'} \qquad \int_0^1 \dots \to I$$

$\boxed{\vartheta < \frac{1}{2}}$: $\boxed{\frac{4\binom{n}{k}\vartheta^{k+1}(1-\vartheta)^{n-k}}{I}}$

MAP odhad

$$\hat{\vartheta} = \frac{k+1}{n+1}$$

max: $g(\vartheta) = \vartheta^{k+1}(1-\vartheta)^{n-k}$

pro $\vartheta \in [0, \frac{1}{2}]$

$$g' = \frac{k+1}{\vartheta}\cdot g - \frac{n-k}{1-\vartheta}\cdot g = 0$$

$$\frac{k+1}{\vartheta} = \frac{n-k}{1-\vartheta} \qquad \frac{1-\vartheta}{\vartheta} = \frac{n-k}{k+1} \qquad \frac{1}{\vartheta} = \frac{n-k}{k+1} + 1 = \frac{n+1}{k+1}$$

$\frac{k+1}{n+1}$ pokud $\frac{k+1}{n+1} < \frac{1}{2}$, jinak $\frac{1}{2}$

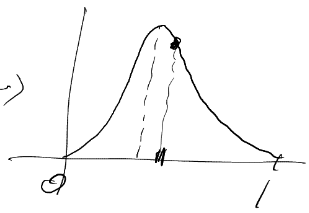
$$\max_{\vartheta \in \{\frac{1}{2}, 1\}} \quad g_2(\vartheta) = \vartheta^k (1-\vartheta)^{n+1-k}$$

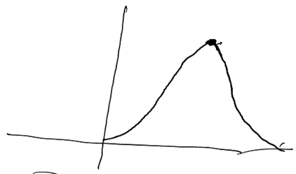
$$\vartheta_{max} = \frac{k}{n+1} \quad \text{pokud } \frac{k}{n+1} > \frac{1}{2}$$

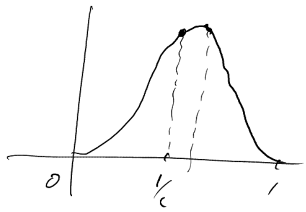
$$\text{jinak } \vartheta_{max} = \frac{1}{2}$$

$g_2'(\vartheta) \ldots$

$$\text{MAP } \hat{\vartheta}: \begin{cases} \frac{k+1}{n+1} & \text{pokud } \frac{k+1}{n+1} < \frac{1}{2} \\ \frac{k}{n+1} & \text{pokud } \frac{k}{n+1} > \frac{1}{2} \\ \frac{1}{2} & \text{jinak} \end{cases}$$







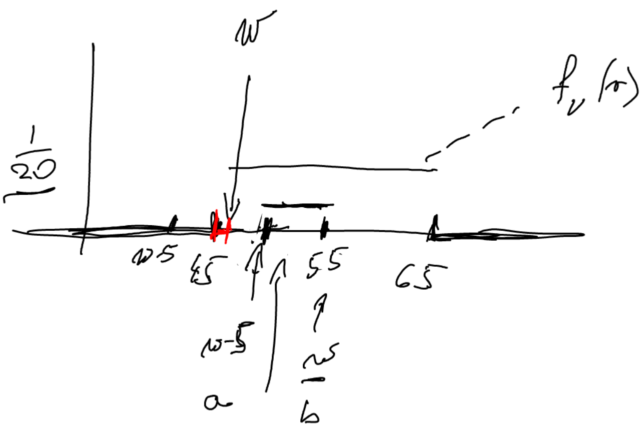
(5) $V \sim U(45, 65)$, $U \sim U(0, 5)$

změříme $V + U =: W$

$$f_{V|W}(v|w) = \frac{\overbrace{f_V(v) \cdot f_{W|V}(w|v)}^{\boxed{g(v)}}}{\int_{45}^{65} f_V(v') f_{W|V}(w|v')\, dv'} \quad \ldots \quad \int_{45}^{65} g(v')\, dv'$$

$$f_{W|V}(w|v) = \begin{cases} \frac{1}{5} & \text{pokud } v < w < v+5 \\ 0 & \text{jinak} \end{cases}$$

$$\text{čitatel } g(v) = \begin{cases} 0 & v < 45 \text{ nebo } v > 65 \\ 0 & v > w \text{ nebo } v < w-5 \\ \frac{1}{100} & \text{pro } v \in [w-5, w] \cap [45, 65] \end{cases}$$



$g(v) = \frac{1}{100}$ na int $[a, b]$

$a = \max(45, w-5)$

$b = \min(65, w)$

$$f_{V|W}(v|w) = \frac{\frac{1}{100}}{\frac{5}{100}} = \frac{1}{5} \text{ na } [a, b]$$

$= 0$ jinak

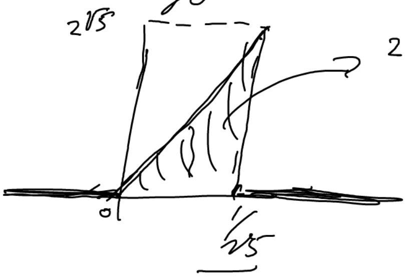
$\Rightarrow E[V / W = w] = w - 0.5$

$$E[V/W] = \frac{a+b}{2}$$

7 a) $T \sim Exp(\vartheta)$ $\quad P(T \geq t) = e^{-\vartheta t}$ $\quad \Theta \dots f_\Theta(\vartheta) = 10\vartheta$

$$f_{T|\Theta}(t|\vartheta) = f_T(t) = \vartheta e^{-\vartheta t}$$

$$f_{\Theta|T}(\vartheta|t) = \frac{f_\Theta(\vartheta)\cdot f_{T|\Theta}(t|\Theta)}{\int \dots} =: g(\vartheta)$$



$2\sqrt{5}\cdot\frac{1}{\sqrt{5}}\cdot\frac{1}{2} = 1$

$$g(\vartheta) = f_\Theta(\vartheta)\cdot f_{T|\Theta}(t|\vartheta) = \begin{cases} 10\vartheta\cdot\vartheta e^{-\vartheta t} & \vartheta \in [0, \frac{1}{\sqrt{5}}] \\ 0 & \vartheta \notin [0, \frac{1}{\sqrt{5}}] \end{cases}$$

$\boxed{\text{MAP}}$ max $g(\vartheta)$, dáno $t = 30$ $\quad \vartheta^2 e^{-30\vartheta}$

$$2\vartheta e^{-30\vartheta} - 30\vartheta^2 e^{-30\vartheta} = 0$$

$$2\vartheta = 30\vartheta^2 \Rightarrow \hat{\vartheta} = \frac{1}{15}$$

podm. stř. hodn. $I = \int_0^{\frac{1}{\sqrt{5}}} 10\vartheta^2 e^{-30\vartheta} = \dots = 7.4\cdot 10^{-4}$

$$E[\Theta|T=30] = \int_0^{\frac{1}{\sqrt{5}}} \vartheta\cdot f_{\Theta|T}(\vartheta|30) = \int_0^{\frac{1}{\sqrt{5}}} \frac{10\vartheta^3 e^{-30\vartheta}}{I} \doteq 0.1$$