Analytic combinatorics
Lecture 11

May 26, 2021

A simple estimate

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Proposition

Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with non-negative coefficients and with radius of convergence $\rho \in (0, +\infty]$. Then for $r \in (0, \rho)$ and $n \in \mathbb{N}_0$, the following holds:

- *If $r \leq 1$, then $a_0 + a_1 + \cdots + a_n \leq \frac{f(r)}{r^n}$.*
- *If $r \geq 1$, then $a_n + a_{n+1} + a_{n+2} + \cdots \leq \frac{f(r)}{r^n}$.*
- *For any $r \in (0, \rho)$, $a_n \leq \frac{f(r)}{r^n}$.*

Proof: $\frac{f(r)}{r^n} = \frac{a_0}{r^n} + \frac{a_1}{r^{n-1}} + \ldots + a_n + a_{n+1} r + \ldots$

$r \leq 1$ — " — $\geq a_0 + a_1 + \ldots + a_n \geq a_n$

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Note: With f, ρ and n be as above, the function $\frac{f(r)}{r^n}$ has a minimum in a point satisfying $\underline{rf'(r) = nf(r)}$.

$$\left(\frac{f(r)}{r^n} \right)' = \frac{f'(r) \cdot r^n - f(r) \cdot n \cdot r^{n-1}}{r^{2n}}$$

Some examples

Examples applying $a_n \le \frac{f(r)}{r^n}$, with $rf'(r) = nf(r)$.

Example 1. Consider $f(z) = e^z$, i.e., $a_n = \frac{1}{n!}$.

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \rho = +\infty, \quad \forall r: \frac{1}{n!} \le \frac{e^r}{r^n}$$

$$r \cdot (e^r)' = n \cdot e^r \Rightarrow r = n$$

$$\frac{1}{n!} \le \frac{e^n}{n^n} = \left(\frac{e}{n}\right)^n, \text{ hence } n! \ge \left(\frac{n}{e}\right)^n$$

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Example 1. Consider $f(z) = e^z$, i.e., $a_n = \frac{1}{n!}$.

Example 2. Estimate $\binom{m}{n}$, with $\underbrace{n \leq m/2}$. $\quad n, m \in \mathbb{N}$

$$f(z) = (1+z)^m = \sum_{n=0}^{m} \binom{m}{n} z^n, \quad \rho = +\infty$$

$$\binom{m}{n} \leq \frac{(1+r)^m}{r^n}; \quad r\left(m(1+r)^{m-1}\right) = n(1+r)^m$$

$$\Rightarrow r \cdot m = n(1+r) \Rightarrow r = \frac{n}{m-n} \leq 1$$

$$\binom{m}{n} \leq \frac{\left(1+\frac{n}{m-n}\right)^m}{\left(\frac{n}{m-n}\right)^n} = \frac{m^m}{\underbrace{n^n (m-n)^{m-n}}} = *$$

$$\binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{n} \leq \text{— '' —}$$

$$\alpha := \frac{m}{n} \rightarrow * = \frac{(\alpha n)^{\alpha n}}{n^n \left((\alpha-1)n\right)^{(\alpha-1)n}} = \frac{\alpha^{\alpha n} n^{\alpha n}}{n^{\alpha n} (\alpha-1)^{(\alpha-1)n}} =$$

$$= \dots \leq (e\alpha)^n = \left(\frac{em}{n}\right)^n \quad \Big| \quad e \geq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1} = \left(1+\frac{1}{\alpha-1}\right)^{\alpha-1}$$

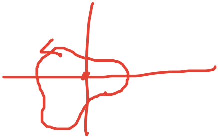
Improving the simple estimate

Recall:

Proposition (Cauchy's integral formula)

Suppose $f = \sum_{n=0}^{\infty} a_n z^n$, with radius of convergence $\rho \in (0, +\infty]$, let γ be the circle of radius $r < \rho$ centered in 0, let $n \in \mathbb{N}_0$. Then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}.$$



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$|z| = r$

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$r \in \gamma$

$$\left| \frac{f(z)}{z^{n+1}} \right| = \left| \cancel{a_n} \sum_{m \ge 0} \frac{a_m z^m}{z^{n+1}} \right| \le \sum_{m \ge 0} \left| \frac{a_m z^m}{z^{n+1}} \right| = \underbrace{\frac{f(r)}{r^{n+1}}}$$

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This is identical to the bound we already know. But we can find better estimates for the integral.

Digression: the landscape of an analytic function

Let $\Omega \subseteq \mathbb{C}$ be a domain, let $f\colon \Omega \to \mathbb{C}$ be an analytic function which is not constant on Ω. What can we say about the function $m\colon \Omega \to [0, +\infty)$ defined as $m(z) = |f(z)|$?

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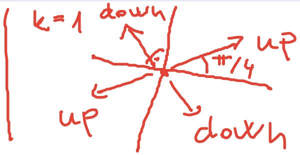
Proposition

Let f, m and Ω be as above. Let $z_0 \in \Omega$ be arbitrary, let $\varepsilon > 0$ be small enough so that $\mathcal{N}_{\leq \varepsilon}(z_0) \subseteq \Omega$. Let $z = z_0 + re^{i\phi}$, with $r \in [0, \varepsilon)$, $\phi \in [0, 2\pi)$.

- *If z_0 is a generic point, then there are constants $\lambda > 0$ and $\tau \in [0, 2\pi)$ such that $m(z) = m(z_0)(1 + \lambda r \cos(\phi - \tau) + O(r^2))$ as $r \to 0$.*
- *It z_0 is a sadle point of multiplicity $k \geq 1$, then there are constants $\lambda > 0$ and $\tau \in [0, 2\pi)$ such that $m(z) = m(z_0)(1 + \lambda r^{k+1} \cos(\underline{(k+1)}\phi - \tau) + O(r^{k+1}))$ as $r \to 0$.*



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In particular, m has no local maxima, and the only local minima satisfy $m(z_0) = 0$.

Proof of the proposition

$$\sqrt{1+\varepsilon} = 1 + \binom{1/2}{1}\varepsilon + \binom{1/2}{2}\varepsilon^2 = 1 + \frac{\varepsilon}{2} + O(\varepsilon^2)$$

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$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \ldots$$

$$= f(z_0) + f'(z_0) r e^{i\phi} + \frac{f''(z_0)}{2} r^2 e^{i2\phi} + \ldots$$

$$m(z) = |f(z)| = |f(z_0)| \cdot \left|\left(1 + \frac{f'(z_0)}{f(z_0)} r e^{i\phi}\right)\right| + O(r^2)$$

$$= m(z_0) \cdot \left|1 + \lambda e^{-i\tau} \cdot r e^{i\phi}\right| + O(r^2)$$

$$= m(z_0) \left|1 + \lambda r \cos(\phi - \tau) + i \lambda r \sin(\phi - \tau)\right| + O(r^2)$$

$$= m(z_0) \sqrt{1 + 2\lambda r \cos(\phi - \tau) + \lambda^2 r^2} = m(z_0)(1 + \lambda r \cos -)$$

Proof continued ...

Back to coefficient estimates

Let us assume (again) that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with nonnegative coefficients and radius of convergence $\rho \in (0, +\infty]$. Recall that

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$$\begin{aligned} a_n &= \frac{1}{2\pi} \left| \int_\gamma \frac{f(z)}{z^{n+1}} \right| \\ &= \frac{1}{2\pi} \left| \int_a^b \frac{f(p(t))}{p(t)^{n+1}} p'(t) \mathrm{d}t \right| \\ &\leq \frac{1}{2\pi} \int_a^b \left| \frac{f(p(t))}{p(t)^{n+1}} \right| \cdot \left| p'(t) \right| \mathrm{d}t \end{aligned}$$

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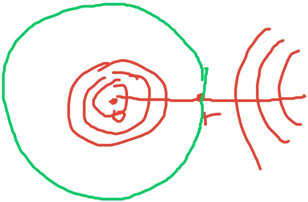

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\end{aligned}$$

$$z = p(t)$$

$$\frac{|f(z)|}{|z^{n+1}|}$$

Ideas:

- We may choose γ so that it passes through (or near) saddle points of $\left| \frac{f(z)}{z^{n+1}} \right|$, so that the maximum of the integrand is small.

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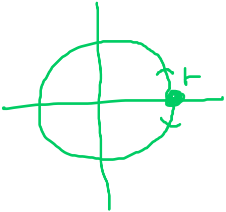

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- We may choose γ so that it passes through (or near) saddle points of $\left| \frac{f(z)}{z^{n+1}} \right|$, so that the maximum of the integrand is small.
- Often $\left| \frac{f(z)}{z^{n+1}} \right|$ is only large in small neighborhoods of the saddle points and very small elsewhere. We may distinguish "large" and "small" regions and bound them separately.

Factorial revisited

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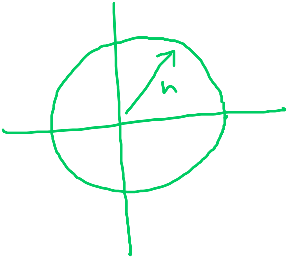
$r \rightarrow$ minimizes $\dfrac{f(r)}{r^n}$

$f(r) = e^r$



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&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\exp(ne^{it})}{n^{n+1}\cancel{e^{(n+1)it}}}\, i\cancel{ne^{it}}\right| \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(\Re(ne^{it}))}{n^n}\,\mathrm{d}t \qquad \text{since } |\exp(w)| = \exp(\Re(w))
\end{aligned}$$

Example: find a better lower bound for $n!$ than $\left(\frac{n}{e}\right)^n$.
Let $\gamma = \gamma(n)$ be a circle around the origin with radius n.
Parametrize γ by $p\colon [-\pi, \pi]$, $p(t) = ne^{it}$.
We have

$$\frac{1}{n!} \le \left(\frac{e}{n}\right)^n$$

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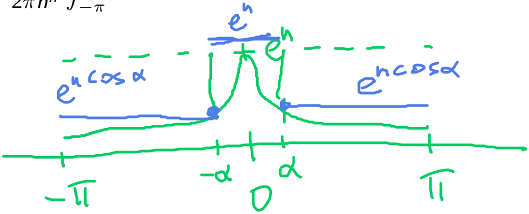
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \frac{\exp(ne^{it})}{n^{n+1}e^{(n+1)it}} ine^{it} \right| \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(\Re(ne^{it}))}{n^n}\mathrm{d}t && \text{since } |\exp(w)| = \exp(\Re(w)) \\
&= \frac{1}{2\pi n^n}\int_{-\pi}^{\pi} \exp(n\cos t)\mathrm{d}t.
\end{aligned}$$



Factorial revisited

Example: find a better lower bound for $n!$ than $\left(\frac{n}{e}\right)^n$.
Let $\gamma = \gamma(n)$ be a circle around the origin with radius n.
Parametrize γ by $p\colon [-\pi, \pi]$, $p(t) = ne^{it}$.
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$$\begin{aligned}
\frac{1}{n!} = [z^n]e^z &= \frac{1}{2\pi i}\int_\gamma \frac{e^z}{z^{n+1}} \\
&= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\exp(ne^{it})}{n^{n+1}e^{(n+1)it}}\, ine^{it}\mathrm{d}t \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\exp(ne^{it})}{n^{n+1}e^{(n+1)it}}\, ine^{it}\right| \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(\Re(ne^{it}))}{n^n}\mathrm{d}t && \text{since } |\exp(w)| = \exp(\Re(w)) \\
&= \frac{1}{2\pi n^n}\int_{-\pi}^{\pi} \exp(n\cos t)\mathrm{d}t.
\end{aligned}$$

Let $\alpha = \alpha(n) \in [0, \pi]$ be a value to be specified later. We will decompose the integral into three integrals over the intervals $[-\pi, -\alpha]$, $[-\alpha, \alpha]$, and $[\alpha, \pi]$. Using trivial bounds on each of the three intervals yields

$(\pi - \alpha)$

$$\int_{-\pi}^{-\alpha} \exp(n\cos t)\mathrm{d}t = \int_{\alpha}^{\pi} \exp(n\cos t)\mathrm{d}t \leq \pi \cdot e^{n\cos\alpha}$$

$$\int_{-\alpha}^{\alpha} \exp(n\cos t)\mathrm{d}t \leq 2\alpha e^n.$$

Finishing the factorial bound

We saw that for any $\alpha \in [0, \pi]$, we have the bound

$\left(\frac{e}{n}\right)^n$ $\qquad$ $\alpha \to 0$

$$\frac{1}{n!} \leq \frac{1}{2\pi n^n}\left(2\pi e^{n\cos\alpha} + 2\alpha e^n\right) = \frac{e^n}{2\pi n^n}\underbrace{\left(2\pi e^{n(\cos(\alpha)-1)} + 2\alpha\right)}_{\to 0 \text{ as } n \to \infty}$$

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We know that for $\alpha \to 0$ we have the Taylor approximation $\cos\alpha = 1 - \frac{\alpha^2}{2} + \cancel{O(\alpha^4)}$.

$$\approx e^{n\left(-\frac{\alpha^2}{2}\right)} \to 0 \iff n\alpha^2 \to +\infty \Rightarrow \alpha \gg \frac{1}{\sqrt{n}}$$

$$\alpha \to 0$$

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We know that for $\alpha \to 0$ we have the Taylor approximation $\cos\alpha = 1 - \frac{\alpha^2}{2} + O(\alpha^4)$. Fix $\alpha = \frac{n^{0.00001}}{\sqrt{n}}$. Then

$$\begin{aligned}\frac{1}{n!} &\leq \frac{e^n}{2\pi n^n} \left(2\pi e^{\left(-2n^{0.00002} + O(n^{-0.99996})\right)} + \frac{2n^{0.00001}}{\sqrt{n}}\right) \\ &\leq O\left(\frac{n^{0.00001}}{\sqrt{n}} \left(\frac{e}{n}\right)^n\right).\end{aligned}$$

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Hence

$$n! \geq \Omega\left(\frac{\sqrt{n}}{n^{0.00001}}\left(\frac{n}{e}\right)^n\right).$$

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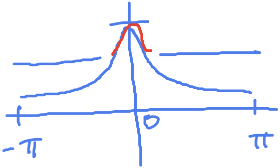


Remark: Stirling approximation gives

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n)).$$

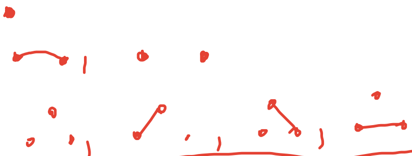
Partial matchings

A partial matching is a graph whose every component is an isolated vertex or an edge. Let p_n be the number of partial matchings on the vertex set $[n]$. Find a bound for p_n.

$p_1 = 1$

$p_2 = 2$

$p_3 = 4$



$$P(z) = \sum_{n=0}^{\infty} p_n \frac{z^n}{n!}$$

$C(z) :=$ EGF of connected matchings (nonempty)

$$C(z) = z + \frac{z^2}{2!} = z + \frac{z^2}{2}$$

$$P(z) = \sum_{k=0}^{\infty} \frac{C^k(z)}{k!} = \exp(C(z)) = e^{z + \frac{z^2}{2}}$$