6. Let $Y = \{Y(x) : x \in \mathbb{R}^d\}$ be a weakly stationary Gaussian random field with the mean value μ and the autocovariance function $C(x,y) = \sigma^2 r(x-y)$, where σ^2 denotes the variance and r is the autocorrelation function of the random field Y. Consider the random measure

$$\Psi(B) = \int_B e^{Y(x)}\, dx, \quad B \in \mathcal{B}^d.$$

The Cox point process Φ with the driving measure Ψ is called a *log-Gaussian Cox process*. Show that the distribution of Φ is determined by its intensity and its pair-correlation function.

recall: $Z \sim N(\mu, \sigma^2) \Rightarrow \mathbb{E} e^{Z} = \exp\{\mu + \frac{\sigma^2}{2}\}$

$\mathbb{E} e^{itZ} = \exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}$

intensity? $\quad \mathbb{E}\Phi(B) = \dots = \lambda \cdot |B|$

$B \in \mathcal{B}^d$

$\|$

$\mathbb{E}[\mathbb{E}[\Phi(B) \mid \Psi]] = \mathbb{E}\Psi(B) = \mathbb{E}\int_B \underbrace{e^{Y(x)}}_{\geq 0}\, dx \overset{\text{Fubini}}{=}$

$= \int_B (\mathbb{E} e^{Y(x)})\, dx = \int_B \exp\{\mu + \frac{\sigma^2}{2}\}\, dx = \exp\{\mu + \frac{\sigma^2}{2}$

$Y(x) \sim N(\mu, \sigma^2)$

$\Rightarrow \lambda = \exp\{\mu + \frac{\sigma^2}{2}\}$

$g(x,y) = \frac{\lambda^{(2)}(x,y)}{\lambda(x)\lambda(y)} = \frac{\lambda^{(2)}(x,y)}{\lambda^2}$ $\qquad \lambda^{(2)}(x,y)$ is density of $\alpha^{(2)}$ w.r.t. Lebesgue meas

$\to E^{[2]}$

A, B disjoint : $\alpha^{(2)}(A \times B) = \mathbb{E}\Phi(A)\Phi(B) = \mathbb{E}\Psi(A)\Psi(B) =$

generate $\mathcal{B}^{[2]}$ $\qquad$ conditioning

$= \mathbb{E}\int_A\int_B e^{Y(x)} e^{Y(y)}\, dx\, dy \overset{\text{Fubini}}{=} \int_A\int_B (\mathbb{E} e^{Y(x)} e^{Y(y)})\, dx\, dy = \dots$

$\dots = \int_A\int_B \lambda^{(2)}(x,y)\, dx\, dy$

$\lambda^{(2)}(x,y) = \mathbb{E} e^{Y(x)} e^{Y(y)} = \mathbb{E} e^{Y(x)+Y(y)} = \exp\{2\mu + \frac{1}{2}\cdot 2\sigma^2(1 + r(x-y))\}$

$x, y \in \mathbb{R}^d$

$(Y(x), Y(y)) \cdot \binom{1}{1} = Y(x) + Y(y) \sim N(2\mu, 2\sigma^2(1 + r(x-y)))$

$\mathrm{var}(Y(x) + Y(y)) = \mathrm{var}\, Y(x) + \mathrm{var}\, Y(y) + 2\,\mathrm{cov}$

$= \sigma^2 + \sigma^2 + 2\cdot\sigma^2 r(x-y) = 2\sigma^2($

$g(x,y) = \frac{\lambda^{(2)}(x,y)}{\lambda^2} = \exp\{\sigma^2 \underbrace{r(x-y)}_{\geq 0}\} \geq 1$ $\quad$ ... clustering ?

$g(r; \sigma^2, \beta)$

$r(\|x-y\|)$

$g(x,y)$ gives σ^2: $\sigma^2 = \log g(x,x)$

r: $r(x-y) = \frac{1}{\sigma^2} \log g(x,y)$

λ gives μ: $\mu = \log \lambda - \frac{\sigma^2}{2}$

determine distri
of $\{Y(x), x \in \mathbb{R}^d$
and hence

Φ is simple point process... Ψ is a diffuse measure

$$\mathbb{P}(\Phi(B)=0) = \mathbb{E}\left[\mathbb{E}[\mathbb{1}(\Phi(B)=0) \mid \Psi]\right] = \mathbb{E}\left[e^{-\Psi(B)}\right] =$$

$$= \mathbb{E} \exp\left\{-\int_B e^{Y(x)}\,dx\right\}$$

.... determined uniquely by μ, σ^2, r

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model fitting? assume $\hat{g}(x,y)$ is available (kernel smoo

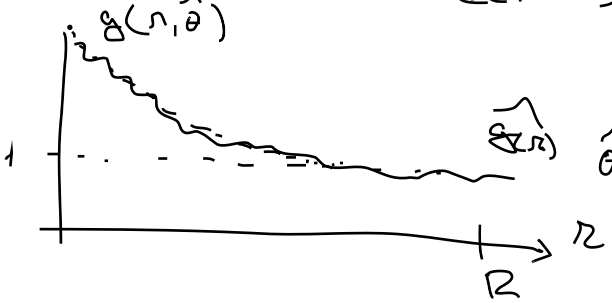
under stationarity : $g(x,y) = g(x-y)$

+ isotropy : $g(x,y) = g(\|x-y\|) - g$

$\Rightarrow$ same for $\hat{g}(x,y)$ ... $\underline{\widehat{g(r)}}$, $r>0$

assume we have a model such that $g(r) = g(r; \underline{\theta})$

$\underline{\theta}$ ... vector of param



$$\hat{\theta} = \underset{\theta}{\arg\min} \int_0^R \left(\widehat{g(r)}^q - g(r;\theta)^q\right)^2 dr$$

$q = \frac{1}{4}, \frac{1}{2}$

$\sum \quad r_1, \dots, r_N$

Minimum contrast estimation