$$N_t = \mathbb{1}[X \le t] \cdot \mathbb{1}[\delta = 1] \quad \left(= \mathbb{1}[\tau \le t] \cdot \mathbb{1}[\delta = 1] \right)$$

čas události

$\delta = 1$ pokud $\tau \le U$ (čas cenzorování)

τ je absolutně spojitá

intenzita rizika $\lambda(t) = \dfrac{f(t)}{1 - F(t)} = -\log' S(t)$

$$N_t - \Lambda(t \wedge X)$$

$X = \tau \wedge U$ čas události

$\delta = 1 \quad X = \tau$

$\delta = 0 \quad X = U$

$$\Lambda(t) = \int_0^t \lambda(s)\,ds$$ kumulativní riziko

$\boxed{N_t - \Lambda(t \wedge X)} = N_t - \int_0^t 1[X \geq u]\, \lambda(u)\, du$ je martingal vůči filtraci

$$\mathcal{F}_t = \sigma(N_s, N_s^U, s \leq t)$$

$$N_s^U = 1[X \leq s, \delta = 0]$$

tehdy a jen tehdy, když $\lambda(t) = \lim_{h \to 0+} \frac{1}{h} \frac{P[\tau \in [t, t+h), U \geq t]}{P[\tau \geq t, U \geq t]}$

$$\lambda(t) := \lim_{h \to 0+} \frac{1}{h} \frac{P[\tau \in [t, t+h)]}{P[\tau \geq t]}$$

1) adaptovanost

$$\mathcal{F}_t = \sigma(N_s, N_s^U ; s \le t)$$

N_t je $\mathcal{F}_t$ adapt.

$$A_t = \int_0^t \underbrace{1[X \ge u]}_{\in \mathcal{F}_u \subset \mathcal{F}_t} \lambda(u)\, du$$

$$1[X \ge t] = 1[\tau \ge t] \cdot 1[U \ge t] =$$

$$= 1[N_{t-} = 0] \cdot 1[N^U_{t-} = 0]$$

$\mathcal{F}_t$ - měřitelná náh. veličina

2) integrovatelnost

$$|M_t| = |N_t - A_t| \le |N_t| + |A_t| \le 1 + \int_0^t \underbrace{1[X \ge u]}_{\ge 0} \underbrace{\lambda(u)}_{\ge 0}\, du \qquad \lambda(u) = \frac{f(u)}{1 - F(u)}$$

$$= 1 + \int_0^t 1[\tau \ge u] \cdot 1[U \ge u]\, \lambda(u)\, du \le 1 + \int_0^t 1[\tau \ge u]\, \lambda(u)\, du =$$

$$= 1 + \int_0^t 1[\tau \geq u] \frac{f(u)}{P[\tau \geq u]} \, du$$

$$E|M_t| \leq 1 + \int_0^t \underbrace{\cancel{E\, 1[\tau \geq u]}}_{P[\tau \geq u]} \frac{f(u)}{\cancel{P[\tau \geq u]}} \, du \leq 1 + \int_0^T f(u)\, du = 2$$

3) mart. vlastnost

$$E\left[N_{t+s} - N_t - \int_t^{t+s} 1[X \geq u] \lambda(u) \, du \,\middle|\, \mathcal{F}_t \right] = 0 \quad \text{s.j. ??}$$

$$\int_0^{t+s} 1[X \geq u] \lambda(u) \, du = \int_0^t 1[X \geq u] \lambda(u) \, du + \int_t^{t+s} 1[X \geq u] \lambda(u) \, du$$

$$E[N_{t+s}-N_t \mid \mathcal{F}_t] \overset{s.j.}{=} E\left[\int_t^{t+s} 1[X \geq u]\,\lambda(u)\,du \,\Big|\, \mathcal{F}_t\right]$$

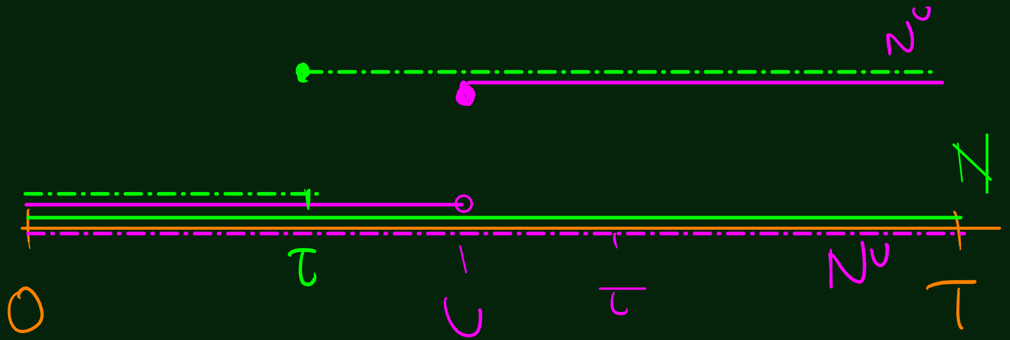
$\forall\, F \in \mathcal{F}_t$

$$\int_F E[N_{t+s}-N_t \mid \mathcal{F}_t]\,dP = \int_F E\left[\int_t^{t+s} 1[X \geq u]\,\lambda(u)\,du \,\Big|\, \mathcal{F}_t\right] dP$$

|| def. || def.

$$\int_F N_{t+s}-N_t\,dP \qquad \int_F \int_t^{t+s} 1[X \geq u]\,\lambda(u)\,du\,dP$$

$\mathcal{F}_t$ je vlastně velmi jednoduchá σ-algebra



$$F = (F \cap [X \leq t]) \cup (F \cap [X > t]) \qquad F \in \mathcal{F}_t$$

a) $\omega \in [X \leq t]$

$N_{t+s} - N_t = 0$ (k událostí došlo před časem t)

$$\int_{F \cap [X \leq t]} N_{t+s} - N_t \, dP = 0$$

$$\int_{F \cap [X \leq t]} \int_t^{t+s} \mathbb{1}[X \geq u] \lambda(u) \, du \, dP$$

$$= \int_F \int_t^{t+s} \underbrace{\mathbb{1}[X \leq t] \, \mathbb{1}[X \geq u]}_{0} \lambda(u) \, du \, dP = 0$$

$$\int_{F\cap[X\le t]} N_{t+s}-N_t \, dP = \int_{F\cap[X\le t]} \int_t^{t+s} 1_{[X\ge u]}\lambda(u)\, du\, dP$$

b) $F\cap[X>t]$ $\quad [X>t]\in\mathcal{F}_t$ $\quad [X>t]$ je atomem $\mathcal{F}_t$, tj. žádná vlastní podmnožina $[X>t)$ jiná než $\emptyset$ a $[X>t)$ není v $\mathcal{F}_t$

Pokud $F\cap[X>t]$ je atomem $\mathcal{F}_t$ (a to je!), pak z konstrukce podmíněné střední hodnoty plyne

pro nějakou konstantu

$$E[N_{t+s}-N_t \mid \mathcal{F}_t](\omega) = k \quad \forall\, \omega\in[X>t]$$

a take

$$E\left[\int_t^{t+s} 1[X\geq u]\,\lambda(u)\,du \,\middle|\, \mathcal{F}_t\right] = k^* \qquad \forall\, \omega\in[X>t]$$

$$\int_{[X>t]} N_{t+s}-N_t \, dP = E\Big[1[X>t]\underbrace{(N_{t+s}-N_t)}_{\in\{0,1\}}\Big] = P[X>t,\ N_{t+s}-N_t=1]$$

$$= P[X\in(t,t+s],\ \delta=1]$$

$$\| $$

$$\int_{[X>t]} \underbrace{E[N_{t+s}-N_t \mid \mathcal{F}_t]}_{=k}\, dP$$

$$\|$$

$$k\cdot P[X>t]$$

$$\Longrightarrow k = \frac{P[t<X\leq t+s,\ \delta=1]}{P[X>t]} = \frac{P[t<T\leq t+s,\ \delta=1,\ U>t]}{P[T>t,\ U>t]}$$

$$E\left[\int_t^{t+\Delta} 1[X\geq u]\,\lambda(u)\,du \,\Big|\, \mathcal{F}_t\right] = k^* \quad \text{pro } \omega\in[X>t]$$

$$k^*\cdot P[X>t] = \int_{[X>t]} E\left[\int_t^{t+\Delta} 1[X\geq u]\,\lambda(u)\,du \,\Big|\, \mathcal{F}_t\right] dP = \int_{[X>t]} \int_t^{t+\Delta} 1[X\geq u]\,\lambda(u)\,du\,dP$$

$$= \int_\Omega \int_t^{t+\Delta} 1[X\geq u]\,\lambda(u)\,du\,dP \qquad \int_\Omega 1[X\geq u]\,dP = P[X\geq u]$$

$$k^* = \frac{\int_t^{t+\Delta} P(X\geq u)\,\lambda(u)\,du}{P[X>t]}$$

$N_t - A_t$ je martingal $\Longleftrightarrow$ $\ell = \ell^*$

$$P[t < \tau \le t+s, U > t, \delta = 1] = \int_t^{t+s} P[\tau \ge u, U \ge u] \lambda(u)\, du \qquad \forall t, \forall s$$

$t \in [0,T]$

$s \ge 0 \quad t+s \le T$

pro malá s a skoro všechna t

$$P[\tau \ge t, U \ge t] \lambda(t) \cdot s + o(s) = P[t < \tau \le t+s, U > t, \delta = 1]$$

pro malé s je δ s velkou pravděp. $= 1$ ($s \to P \Longrightarrow 1$)

$$\lambda(t) = \lim_{s \to 0+} \frac{P[t \le \tau < t+s, U \ge t]}{s\, P[\tau \ge t, U \ge t]}$$