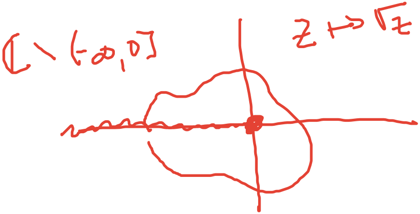
Analytic combinatorics

Lecture 10

May 19, 2021

Non-meromorphic example

Not all functions are meromorphic (e.g. $z \mapsto \sqrt{z}$). Today we will look at the coefficient asymptotics of an important class of non-meromorphic functions.



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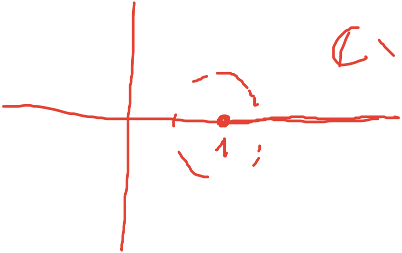
$\mathbb{C} \setminus [1, +\infty)$



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For $\alpha \in \mathbb{Z}_{\leq 0} = \{0, -1, -2, -3, \dots\}$, f_α is a polynomial of degree $-\alpha$, hence its coefficients are eventually 0. From now on, assume $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$.

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Fact (Generalized binomial theorem)

$$f_\alpha(z) = (1-z)^{-\alpha} = \sum_{n=0}^\infty \binom{-\alpha}{n}(-1)^n z^n,$$

where for $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$ we define

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Goal: Figure out the asymptotics of $\binom{-\alpha}{n}(-1)^n$ as $n \to \infty$.

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Observe: For $\alpha \in \mathbb{N}$, $f_\alpha(z) = \frac{1}{(1-z)^\alpha}$ is actually meromorphic, and we have

$$\begin{aligned}
\binom{n-1+\alpha}{n} &= \binom{n-1+\alpha}{\alpha-1} \\
&= \frac{(n+\alpha-1)(n+\alpha-2)\cdots(n+1)}{(\alpha-1)!} \\
&= \frac{n^{\alpha-1}}{(\alpha-1)!}\left(1+O\left(\frac{1}{n}\right)\right).
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Can we say something similar for $\alpha \notin \mathbb{Z}$?

Euler's Gamma function

Definition

For a complex number α with $\Re(\alpha) > 0$, define the function

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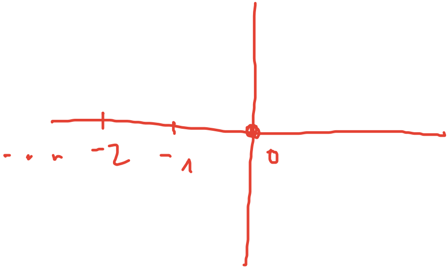
Fact: The above integral converges for any α with $\Re(\alpha) > 0$. The function Γ has an analytic continuation into a meromorphic function on $\mathbb{C}$, with a pole of order 1 in every $m \in \mathbb{Z}_{\leq 0}$ and no other poles.



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Proposition

The function Γ has the following properties:

1. $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} dx = \left[-e^{-x}\right]_0^{+\infty} = 0 - (-1) = 1$$

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$$= (n-1)(n-2)!$$
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4. *For* $n \in \mathbb{N}$ *and* $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$*:* $\binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}$

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5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (no proof)

Proof of the proposition

$$\alpha \in (0,+\infty): \quad \boxed{\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)}$$

$$\Gamma(\alpha+1) = \int_0^{+\infty} x^{\alpha} e^{-x}\,dx \qquad (\text{per partes})$$

$$(x^{\alpha})' = \alpha x^{\alpha-1}, \qquad \int e^{-x}\,dx = -e^{-x}$$

$$\left[x^{\alpha}(-e^{-x})\right]_0^{+\infty} - \int_0^{+\infty} \alpha x^{\alpha-1}(-e^{-x})\,dx =$$

$$= 0 + \alpha \cdot \int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx = \alpha\,\Gamma(\alpha).$$

$$\binom{n+\alpha-1}{n} = \frac{(n+\alpha-1)(n+\alpha-2)\cdot\ldots\cdot(\alpha+1)\cdot\alpha}{n!} = \frac{\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}}{\Gamma(n+1)}$$

$$\Gamma(n+\alpha) = (n+\alpha-1)\,\Gamma(n+\alpha-1) = (n+\alpha-1)(n+\alpha-2)\,\Gamma(n+\alpha-2)$$

$$\ldots = (n+\alpha-1)(n+\alpha-2)\cdot\ldots\cdot(\alpha+1)\cdot\alpha\cdot\Gamma(\alpha)$$

More on the Γ function

Fact (Generalized Stirling approximation)

For $x \in \mathbb{R}$, we have

(x!)

$$\Gamma(x+1) = x\Gamma(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + O(x^{-1})\right) \text{ as } x \to +\infty.$$

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Corollary

Recall that $f_\alpha(z) = \frac{1}{(1-z)^\alpha}$ and that

$$[z^n]f_\alpha(z) = \binom{n+\alpha-1}{n} = \underbrace{\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}}.$$

For $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$, as $n \to +\infty$, we have

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$$\frac{\sqrt{2\pi(n+\alpha-1)} \cdot \left(\frac{n+\alpha-1}{e}\right)^{n+\alpha-1}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \sim (n+\alpha-1)^{\alpha-1} \underbrace{\frac{1}{e^{\alpha-1}}} \underbrace{\left(\frac{n+\alpha-1}{n}\right)^n}$$

$$(n+\alpha-1)^{\alpha-1} \downarrow n^{\alpha-1}\left(1+O\left(\frac{1}{n}\right)\right) \qquad \left(\frac{n+\alpha-1}{n}\right)^n \longrightarrow e^{\alpha-1}$$

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Corollary

Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Define $g(z) = \frac{1}{(1-\gamma z)^\alpha}$. Then

$$[z^n]g(z) = \gamma^n [z^n] f_\alpha(z) = \frac{\gamma^n n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Binary trees

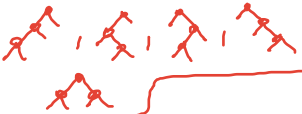
A binary tree is either a single leaf node, or an internal root node together with an ordered pair of subtrees, which are both binary trees. Let t_n be the number of binary trees with n internal nodes. What can we say about the asymptotics of t_n?

$t_0 = 1$ $\bullet$

$t_1 = 1$

$t_2 = 2$

$t_3 = 5$



$$T(z) = \sum_{n=0}^{\infty} t_n z^n, \qquad t_n = \sum_{k=0}^{n-1} t_k t_{n-k-1} \quad (n \geq 1)$$

$$\underbrace{\sum_{n=1}^{\infty} z^n t_n}_{= T(z)-1} = \sum_{n=1}^{\infty} z^n \left(\sum_{k=0}^{n-1} t_k t_{n-k-1} \right) = \sum_{n=1}^{\infty} z \cdot \sum_{k=0}^{n-1} t_k z^k \cdot t_{n-k-1} z^{n-k-1}$$

$$= z \cdot \sum_{n=0}^{\infty} \sum_{k=0}^{n} t_k z^k \cdot t_{n-k} z^{n-k} = z \cdot T^2(z)$$

$$T(z) - 1 = z \cdot T^2(z) \quad \text{or} \quad \boxed{T(z) = 1 + z T^2(z)}$$

$$\mathcal{T} = \{ \underset{\text{size } 0}{\bullet} \} \cup \{ \overset{\text{size } 1}{\text{root}} \} \times \mathcal{T} \times \mathcal{T}$$

$$T(z) = 1 + z\,T^2(z) \iff z\,T^2(z) - T(z) + 1 = 0$$

$$\Rightarrow \frac{1 \pm \sqrt{1-4z}}{2z} = T_{1,2}(z), \quad \frac{1+\sqrt{1-4z}}{2z} \text{ is}$$

not analytic in $z=0$, hence $\underbrace{T(z) = \frac{1-\sqrt{1-4z}}{2z}}$

$$t_n = [z^n]\,T(z) = [z^{n+1}]\,\frac{1-\sqrt{1-4z}}{2} = [z^{n+1}]\left(\frac{1}{2} - \frac{1}{2}\sqrt{1-4z}\right)$$

$$= -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} = -\frac{1}{2}\cdot(-4)^{n+1}\cdot\binom{\frac{1}{2}}{n} = \ldots = \frac{1}{n+1}\binom{2n}{n}$$

(Catalan number)

$$\parallel$$

$$-\frac{1}{2}\cdot 4^{n+1}\,[z^{n+1}]\sqrt{1-z} = -\frac{1}{2}\cdot 4^{n+1}\cdot\frac{(n+1)^{-3/2}}{\Gamma(-\frac{1}{2})}\left(1+O\left(\frac{1}{n}\right)\right)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \left(-\tfrac{1}{2}\right)\Gamma\left(-\tfrac{1}{2}\right)$$

$$\parallel$$

$$\sqrt{\pi} \quad \text{so } \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}$$

$$f_{-\frac{1}{2}}(z)$$

$$\parallel$$

$$4^n\,\frac{1}{\sqrt{\pi}\cdot n^{3/2}}\cdot\left(1+O\left(\frac{1}{n}\right)\right)$$

Fact

Let $\rho > 1$, let $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$. Let f be a function defined on $\Omega = \mathcal{N}_{<\rho}(0) \setminus [1, +\infty)$ as

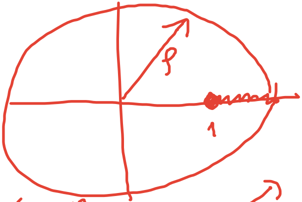
$$f(z) = \frac{g(z)}{(1-z)^\alpha},$$

where $g(z)$ is analytic on $\mathcal{N}_{<\rho}(0)$ and $g(1) \neq 0$. Then

$$[z^n]f(z) = g(1)\left(1 + O\left(\frac{1}{n}\right)\right)[z^n]\frac{1}{(1-z)^\alpha} = \underbrace{\frac{g(1)n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right)}.$$

$$\frac{1}{(1-z)^\alpha} \quad \checkmark$$

Vague intuition



$$g(z) = g(1) + g_1(z-1) + g_2(z-1)^2 + \dots$$

$$\frac{g(z)}{(1-z)^\alpha} = \frac{g(1)}{(1-z)^\alpha} + \frac{-g_1}{(1-z)^{\alpha-1}} + \frac{g_2}{(1-z)^{\alpha-2}} + \dots$$

$$\frac{-g_1 \cdot n^{\alpha-2}}{\Gamma(\alpha-1)}$$

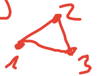
2-regular graphs

Let g_n be the number of 2-regular graphs on the vertex set $[n]$. What can we say about the asymptotics of g_n?

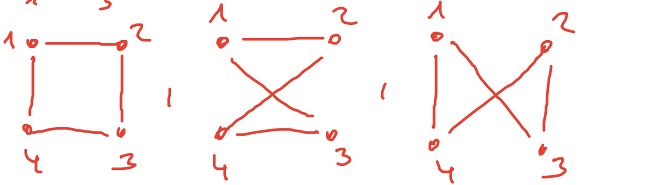
$g_0 = g_1 = g_2 = 0$

$G(z) = \sum_{n=0}^{\infty} \frac{g_n}{n!} z^n$

$g_3 = 1$



$g_4 = 3$



$\vdots$

$c_n :=$ # of cycles on $[n]$ $= \frac{(n-1)!}{2}$ for $n \geq 3$

$$C(z) = \sum_{n=3}^{\infty} \frac{c_n}{n!} z^n = \sum_{n=3}^{\infty} \frac{z^n}{2n} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{z^n}{n} - z - \frac{z^2}{2} \right)$$

$$= \frac{1}{2}\left(\ln\left(\frac{1}{1-z}\right) - z - \frac{z^2}{2} \right);$$

$C^k(z)$ ... EGF of ordered k-tuples ...

$$C(z) = \frac{1}{2}\left(\ln\left(\frac{1}{1-z}\right) - z - \frac{z^2}{2}\right)$$

$$G(z) = \sum_{k=0}^{\infty} \frac{C^k(z)}{k!} = \exp\left(C(z)\right) =$$

$$= \exp\left(\frac{1}{2}\ln\left(\frac{1}{1-z}\right)\right) \cdot \exp\left(-\frac{z}{2} - \frac{z^2}{4}\right)$$

$$= \frac{\exp\left(-\frac{z}{2} - \frac{z^2}{4}\right)}{\sqrt{1-z}}$$