$\tilde{W}^n_t,\ t \in [0,1]$ $\qquad \tilde{W}^n_t = \sum_{j=1}^{\tau_n(t)} \tilde{X}_{n,j}$

$(D[0,1], d_D)$

$$\forall \varepsilon > 0 \quad \lim_{\delta \to 0} \limsup_{n \to \infty} P\Big[\sup_{\substack{|s-t|\le\delta \\ s,t\in[0,1]}} |\tilde{W}^n_t - \tilde{W}^n_s| > \varepsilon\Big] = 0 \qquad ?$$

<u>Lemma:</u> Nechť $U = \{U_i,\ i = 1, 2, \dots\}$ je martingal, $U_0 = 0$

Pak pro libovolné $c > 0$ platí

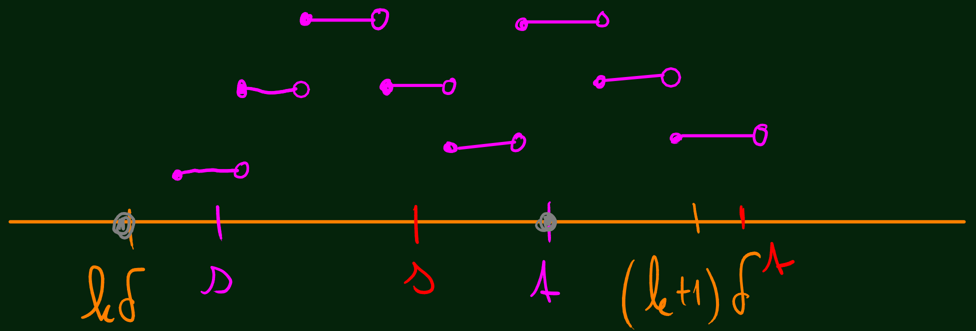
$$P\Big[\max_{0\le j\le n} |U_j| > 2c\Big] \le \frac{2}{c}\, E\big[|U_n| \cdot \mathbb{1}_{|U_n| \ge c}\big]$$

$\delta > 0$ pevne zvolené $s < t$

$$P\Big[\sup_{|t-s|\le\delta} |\tilde W_t^n - \tilde W_s^n| > \varepsilon\Big] = P\Big[\sup_{|t-s|\le\delta} \Big|\sum_{j=r_n(s)+1}^{r_n(t)} \tilde X_{n,j}\Big| > \varepsilon\Big]$$

$$\sum_{j=1}^{r_n(t)} \tilde X_{n,j} \qquad \sum_{j=1}^{r_n(s)} \tilde X_{n,j}$$

$$\le \sum_{j=0}^{\lfloor 1/\delta \rfloor} P\Big[\sup_{j\delta \le t \le (j+1)\delta} \Big|\sum_{k=r_n(j\delta)}^{r_n(t)} \tilde X_{n,k}\Big| > \frac{\varepsilon}{3}\Big]$$



$2c = \frac{\varepsilon}{3}$ a použijeme lemma

$$\le \frac{12}{\varepsilon} \sum_{j=0}^{\lfloor 1/\delta \rfloor} E\Big(\Big|\sum_{k=r_n(j\delta)}^{r_n((j+1)\delta)} \tilde X_{n,k}\Big| \cdot 1\Big[\Big|\sum_k \tilde X_{n,k}\Big| > \frac{\varepsilon}{6}\Big]\Big)$$

$$\leq \frac{12}{\varepsilon} \sum_{j=0}^{\lfloor 1/\delta \rfloor} \left(E\left(\sum_{k=n_m(j\delta)}^{n_m((j+1)\delta)} \tilde{X}_{m,k} \right)^2 \right)^{1/2} \cdot \Bigg(\underbrace{P\left(\left| \sum_{k=n_m(j\delta)}^{n_m((j+1)\delta)} \tilde{X}_{m,k} \right| \geq \frac{\varepsilon}{6} \right)}_{\to\, 2\left(1-\Phi\left(\frac{\varepsilon}{6\sqrt{\delta}}\right)\right)} \Bigg)^{1/2}$$

Φ d.f. $N(0,1)$

nezávisí na s a t

$$\sum_{k=n_m(j\delta)}^{n_m((j+1)\delta)} \tilde{X}_{m,k} = \tilde{W}^n_{(j+1)\delta} - \tilde{W}^n_{j\delta} \xrightarrow{d} N(0,\delta)$$

δ pevné, $n \to \infty$

$$E\,\tilde{X}_{m,k}\tilde{X}_{m,l} = 0 \quad k \neq l$$

$$E\left[\tilde{X}_{m,l} \cdot \underbrace{E(\tilde{X}_{m,k} \mid \mathcal{F}_{m,l})}\right] = 0 \quad k > l$$

$$E\left(\sum \tilde{X}_{m,k}\right)^2 = E \sum_{k=n_m(j\delta)}^{n_m((j+1)\delta)} \tilde{X}_{m,k}^2 \xrightarrow{P} (j+1)\delta - j\delta = \delta$$

$= 0$

Pro $\delta > 0$ pevné a pro $n \to \infty$

$$\frac{12}{\varepsilon}\sum_{j=0}^{\lfloor \frac{1}{\delta} \rfloor} \left(E\left(\sum_{k=n_n(j\delta)} X_{n,k} \right)^2 \right)^{1/2} \cdot \left(P\left(\left| \sum_{n_n(j\delta)}^{n_n((j+1)\delta)} \tilde{X}_{n,k} \right| > \frac{\varepsilon}{6} \right) \right)^{1/2} \longrightarrow$$

$$\to \frac{12}{\varepsilon}\sum_{j=0}^{\lfloor \frac{1}{\delta} \rfloor} \delta^{1/2} \cdot \left[2\left(1 - \Phi\left(\frac{\varepsilon}{6\sqrt{\delta}}\right)\right) \right]^{1/2} \leq \frac{12\sqrt{2}}{\varepsilon} \cdot \frac{2}{\delta^{1/2}} \cancel{\delta^{1/2}} \left(1 - \Phi\left(\frac{\varepsilon}{6\sqrt{\delta}}\right)\right)^{1/2}$$

$$\left\lfloor \frac{1}{\delta} \right\rfloor \leq \frac{2}{\delta}$$

$$\propto \left(\frac{1 - \Phi\left(\frac{\varepsilon}{6\sqrt{\delta}}\right)}{\delta} \right)^{1/2}$$

pro $\delta \to 0$ máme limitu typu „$\frac{0}{0}$"

$$\frac{1-\Phi\left(\frac{\varepsilon}{6\sqrt{\delta}}\right)}{\delta}$$

l'Hospitalovo pravidlo

$$\frac{-\varphi\left(\frac{\varepsilon}{6\sqrt{\delta}}\right)\cdot\frac{\varepsilon}{6\,\delta^{3/2}}\cdot\left(-\frac{1}{2}\right)}{1}=\frac{\varepsilon}{12\,\delta^{3/2}}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\varepsilon^{2}}{72\delta}\right\}$$

$$C e^{-kx}\cdot x^{3/2}\xrightarrow{x\to\infty}0$$

Ověřili jsme těsnost a tím i dokázali větu.

s.m.d. odpovídající náhodné procházce

$r_n = n$ $\quad X_{1,1} \quad P[X_{1,1}=1] = \frac{1}{2} = P[X_{1,1}=-1]$

$X_{2,1}, X_{2,2}$ iid $\quad P\left[X_{2,1} = \frac{1}{\sqrt{2}}\right] = \frac{1}{2} = P\left[X_{2,1} = -\frac{1}{\sqrt{2}}\right]$

$$\sum_{l=1}^{n} X_{n,l}^2 = 1$$

$\downarrow$

c

$X_{n,i}$ iid $i = 1, \dots, n$

$$P\left[X_{n,i} = \frac{1}{\sqrt{n}}\right] = \frac{1}{2} = P\left[X_{n,i} = -\frac{1}{\sqrt{n}}\right]$$

$$E \max_k X_{n,k}^2 = \frac{1}{n} \to 0$$

$$\left(\sum_{j=1}^{r_n(t)} X_{n,j},\ t \in [0,1] \right) \xrightarrow{d} \left(W_t,\ t \in [0,1] \right)$$