Analytic combinatorics

Lecture 9

May 5, 2021

Cauchy's integral formula

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where $f^{(n)}$ is the n-th derivative of f. But this is seldom useful in practice.

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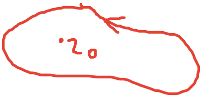
Theorem (Cauchy's integral formula)

Let $\gamma \subseteq \mathbb{C}$ be a simple closed curve, and let $z_0 \in \mathrm{Int}(\gamma)$. Let f be a function analytic on a domain Ω with $\gamma \cup \mathrm{Int}(\gamma) \subseteq \Omega$, and suppose f admits the expansion $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Then, for every $n \geq 0$ we have

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}}.$$



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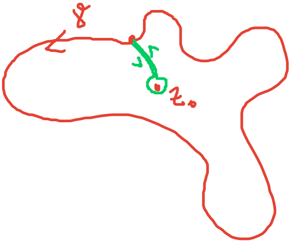
$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}.$$

Consequence: The value $f(z_0)$ (which is equal to a_0) can be determined as $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}$, and in particular, the value of f in z_0 is uniquely determined by its values on γ.

Proof of Cauchy's integral formula



Note: $\dfrac{f(z)}{(z-z_0)^{n+1}}$ is analytic on $\underbrace{\Omega \setminus \{z_0\}}$

WLOG γ is a circle around z_0 of small enough radius, smaller than the radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\frac{f(z)}{(z-z_0)^{n+1}} = \underbrace{\frac{a_0}{(z-z_0)^{n+1}} + \frac{a_1}{(z-z_0)^{n}} + \dots +}_{\text{prim. function on } \Omega\setminus\{z_0\}} \frac{a_n}{z-z_0} + \underbrace{a_{n+1} + a_{n+2}(z-z_0) + \dots}_{\text{analytic on } \gamma \cup \mathrm{Int}(\gamma)}$$

$$\boxed{\text{p.f. of } \frac{1}{z^k}\left[= z^{-k}\right] = \frac{1}{-k+1} z^{-k+1} \quad (k \geq 2)}$$

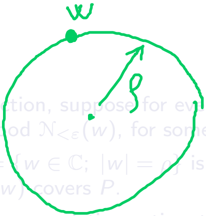
$$\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} = \int_\gamma \frac{a_n}{z-z_0} = 2\pi i a_n$$

$\square$

Consequence of CIF

Theorem (Easy part of Pringsheim's theorem)

Suppose f is analytic in 0, with series exansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose that the series has radius of convergence $\rho \in (0, +\infty)$. Then there is a point $w \in \mathbb{C}$ with $|w| = \rho$ such that f has no analytic continuation to a domain containing w.



For contradiction, suppose for every w with $|w| = \rho$, f has an analytic continuation to a neighborhood $\mathcal{N}_{<\varepsilon}(w)$, for some $\varepsilon = \varepsilon(w) > 0$.

The set $C = \{w \in \mathbb{C};\ |w| = \rho\}$ is compact. Hence it has a finite subset P s.t. $\bigcup_{w \in P} \mathcal{N}_{<\varepsilon}(w)$ covers P.

Hence f has an analytic continuation to $\Omega^+ := \Omega \cup \bigcup_{w \in P} \mathcal{N}_{<\varepsilon}(w)$.

The domain Ω^+ contains a circle γ centered at the origin with radius $R > \rho$. Cauchy:

$$|a_n| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} \right| \leq \frac{1}{2\pi} \cdot \mathrm{len}(\gamma) \cdot \frac{\max_{z \in \gamma} |f(z)|}{R^{n+1}} = \frac{\max_{z \in \gamma} |f(z)|}{R^n}.$$

Hence the exponential growth rate of (a_n) is at most $\frac{1}{R}$, and its radius of convergence is at least $R > \rho$, a contradiction. □

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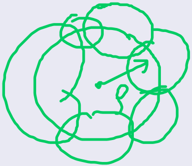

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Residue theorem

Recall: If a function f has a pole (of order d) in a point p, then on some $\mathcal{N}^*_{<\varepsilon}(p)$, we have

$$f(z) = \frac{a_{-d}}{(z-p)^d} + \frac{a_{-d+1}}{(z-p)^{d-1}} + \cdots + \frac{a_{-1}}{z-p} + a_0 + a_1(z-p) + a_2(z-p)^2 + \cdots$$

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Definition

The coefficient a_{-1} in the above expansion is known as the residue of f in p, denoted $\mathrm{Res}_p(f)$. If a function f is analytic in p, we put $\mathrm{Res}_p(f) = 0$.

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Let γ be a closed simple curve, let f be a function meromorphic on a domain Ω containing $\gamma \cup \mathrm{Int}(\gamma)$. Suppose that no pole of f is on γ, and only finitely many poles of f are in $\mathrm{Int}(\gamma)$. Let P be the set of poles of f in $\mathrm{Int}(\gamma)$. Then

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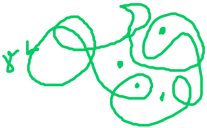

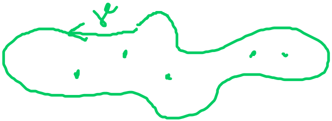

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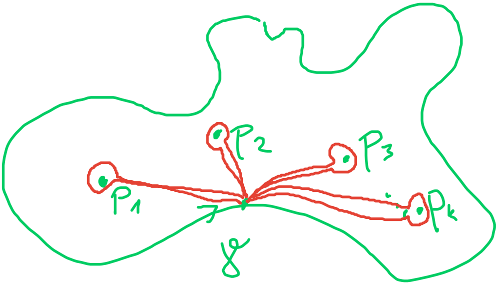
Note: Cauchy's formula is a special case of the Residue theorem, since $\mathrm{Res}_{z_0}\left(\frac{f(z)}{(z-z_0)^{n+1}}\right) = a_n$.

$$\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} = 2\pi i\, a_n$$



around p_j: $f(z) = \underbrace{\frac{a_{-d}}{(z-p_j)^d} + \dots +}_{\text{prim } f} \frac{a_{-1}}{z-p_j} + \underbrace{a_0 + a_1(z-p_j)}_{\text{analytic}}$

$$\int_{\gamma_j} f = \int_{\gamma_j} f = 2\pi i \, a_{-1} = 2\pi i \operatorname{Res}_{p_j}(f)$$

↳ small circle around p_j

$$\int_\gamma f = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{p_j}(f). \qquad \square$$

Bits of theory

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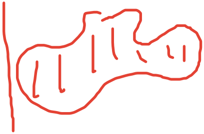
$\mathbb{C} \setminus \{0\}$ is not simply connected



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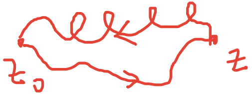
- Recall: For a function f analytic on a simply connected domain Ω and any closed curve $\gamma \subseteq \Omega$, we have $\int_\gamma f = 0$.
- Fact ("Morera's theorem"): Suppose that f is a continuous (not necessarily analytic) function on a (not necessarily simply connected) domain Ω such that for every closed curve $\gamma \subseteq \Omega$ we have $\int_\gamma f = 0$. Then f has a primitive function F on Ω, and in particular f and F are analytic on Ω.

Idea: fix $z_0 \in \Omega$. $F(z) := \int_{\gamma_z} f$, where γ_z is any curve from z_0 to z.



Ordered set partitions re-revisited

Let s_n be the number of ordered set partitions of $[n]$. Here is what we already know:

- $\sum_{n=0}^{\infty} s_n \frac{z^n}{n!} = \frac{1}{2-\exp(z)}$ for $|z| < \ln 2$. Hence the exponential growth rate of $s_n/n!$ is $\frac{1}{\ln 2}$.

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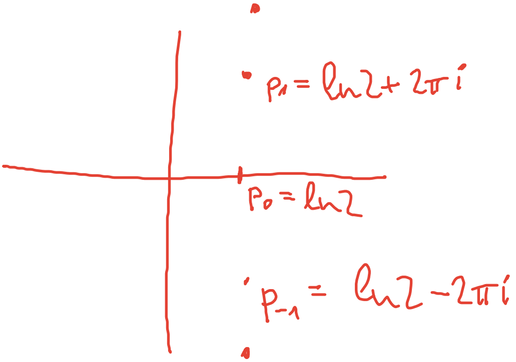

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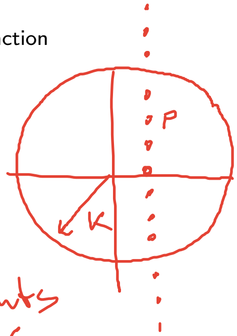
- For any $K > 0$, taking the set $P_K = \{p \in P;\ |p| \leq K\}$, the function

$$f(z) - \sum_{p \in P_K} -\frac{1}{2(z-p)}$$

is analytic on $\mathcal{N}_{<K+\varepsilon}(0)$, and hence

$$\frac{s_n}{n!} = \sum_{p \in P_K} \frac{1}{2p^{n+1}} + O\left(\frac{1}{K^n}\right) \text{ as } n \to \infty.$$



hidden constants depend on K

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$\mathrm{res}_p(f) = -\frac{1}{2}$

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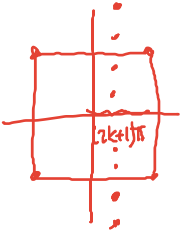

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boundary of the

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Note: $\mathrm{Int}(\gamma_K) \cap P = \{p_j;\ j = -K, -K+1, \dots, K-1, K\}$.

Ordered set partitions – endgame

Recall: We know that

$$\frac{s_n}{n!} - \sum_{j=-K}^{K} \frac{1}{2p_j^{n+1}} = \frac{1}{2\pi i} \int_{\gamma_K} \frac{f(z)}{z^{n+1}}.$$

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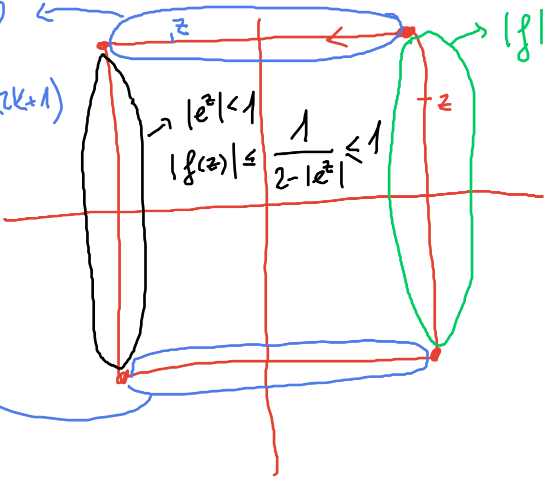
$f(z) = \dfrac{1}{2-e^z}$

We may check that for any K and any $z \in \gamma_K$, $|f(z)| \le 1$ and $|z| \ge (2K+1)\pi$.

$e^z = -e^x < 0$

$z = x + \pi i(2K+1)$, $x \in \mathbb{R}$

$|f(z)| \le \frac{1}{2}$



$|f| \le \dfrac{1}{|e^z|-2} \le$

$\le \dfrac{1}{e^\pi - 2} \le 1$

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$$= 8(2K+1)\pi$$

$$\leq \frac{1}{2\cancel{\pi}} \cdot \mathrm{len}(\gamma_K) \cdot \frac{1}{((2K+1)\pi)^{n+1}}$$

$$\leq \underbrace{\frac{1}{(2\pi K)^n}}_{},$$

$\mathbb{N}$

$$\left| s_n - n! \sum_{j=-K}^{K} \frac{1}{2p_j^{n+1}} \right| \leq \frac{n!}{(2\pi K)^n}$$

which tends to 0 as $K \to \infty$, and for $K = n$ is much smaller than $\frac{1}{n!}$.

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Concluding:

$$\begin{aligned} \left| \frac{s_n}{n!} - \sum_{j=-K}^{K} \frac{1}{2p_j^{n+1}} \right| &= \frac{1}{2\pi} \left| \int_{\gamma_K} \frac{f(z)}{z^{n+1}} \right| \\ &\leq \frac{1}{2\pi} \cdot \operatorname{len}(\gamma_K) \cdot \frac{1}{((2K+1)\pi)^{n+1}} \\ &\leq \frac{1}{(2\pi K)^n}, \end{aligned}$$

which tends to 0 as $K \to \infty$, and for $K = n$ is much smaller than $\frac{1}{n!}$. Hence:

- $$s_n = n! \sum_{j=-\infty}^{\infty} \frac{1}{2p_j^{n+1}}, \text{ and}$$
- s_n is the nearest integer to
  $$n! \sum_{j=-n}^{n} \frac{1}{2p_j^{n+1}}. = n!\,2\left(\frac{1}{\ln 2}\right)^{n+1} + \dots$$