Analytic combinatorics
Lecture 8

April 28, 2021

Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.

The curve is said to be. . .

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

The bounded one is the interior of γ, *denoted* $\mathrm{Int}(\gamma)$, *the other is the exterior of* γ, *denoted* $\mathrm{Ext}(\gamma)$.

Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i \int_a^b \Im(f)$.

The curve is said to be. . .

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$, *the other is the exterior of γ, denoted* $\mathrm{Ext}(\gamma)$.

Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$, *the other is the exterior of γ, denoted* $\mathrm{Ext}(\gamma)$.

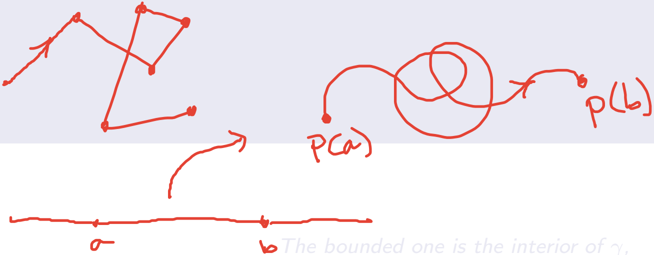
Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The (piecewise smooth) curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.



The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$, *the other is the exterior of γ, denoted* $\mathrm{Ext}(\gamma)$.

Complex integration

What we know:

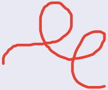
- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The (piecewise smooth) curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.

The curve is said to be...

- simple if p is injective,



The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$, *the other is the exterior of γ, denoted* $\mathrm{Ext}(\gamma)$.

Complex integration

What we know:

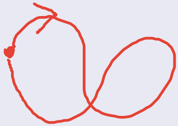
- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The (piecewise smooth) curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.

The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,



The bounded one is the interior of γ, denoted Int(γ), *the other is the exterior of γ, denoted* Ext(γ).

Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The (piecewise smooth) curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.

The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$, *the other is the exterior of* γ, *denoted* $\mathrm{Ext}(\gamma)$.

Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The (piecewise smooth) curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.

The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$, *the other is the exterior of γ, denoted* $\mathrm{Ext}(\gamma)$.

Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The (piecewise smooth) curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.

The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

Fact ("Jordan's curve theorem")

If $\gamma \subseteq \mathbb{C}$ is a simple closed curve, then $\mathbb{C} \setminus \gamma$ is a disjoint union of two domains, one of which is bounded and the other unbounded.

Complex integration

What we know:

- For a real function $f\colon [a,b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.
- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The (piecewise smooth) curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.

The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

Fact ("Jordan's curve theorem")

If $\gamma \subseteq \mathbb{C}$ is a simple closed curve, then $\mathbb{C} \setminus \gamma$ is a disjoint union of two domains, one of which is bounded and the other unbounded. The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$, the other is the exterior of γ, denoted $\mathrm{Ext}(\gamma)$.

Contour integral

Definition

Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p\colon [a, b] \to \mathbb{C}$, let $f : \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

$$\int_\gamma f := \int_a^b f(p(t))p'(t)\mathrm{d}t.$$

Contour integral

Definition

Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p\colon [a,b] \to \mathbb{C}$, let $f\colon \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

$$\int_\gamma f := \int_a^b f(p(t))p'(t)\mathrm{d}t.$$

Fact: The value of the integral does not depend on the choice of the parametrization of γ; it only depends on γ itself, including its orientation.

Definition

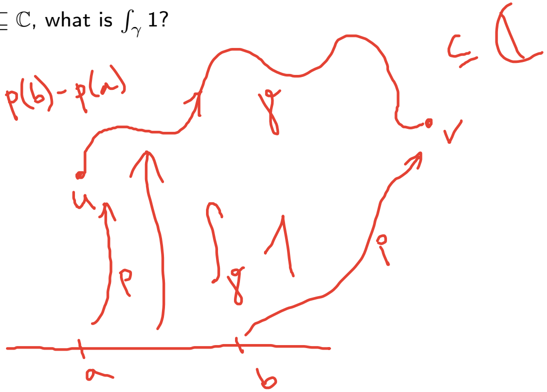
Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p\colon [a, b] \to \mathbb{C}$, let $f : \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

$$\int_\gamma f := \int_a^b f(p(t))p'(t)\mathrm{d}t.$$

Fact: The value of the integral does not depend on the choice of the parametrization of γ; it only depends on γ itself, including its orientation.

Example: For a curve $\gamma \subseteq \mathbb{C}$, what is $\int_\gamma 1$?

$$\int_\gamma 1 = v - u = p(b) - p(a)$$



Definition

Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p\colon [a,b] \to \mathbb{C}$, let $f : \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

$$\int_\gamma f := \int_a^b f(p(t))p'(t)\mathrm{d}t.$$

Fact: The value of the integral does not depend on the choice of the parametrization of γ; it only depends on γ itself, including its orientation.

Example: For a curve $\gamma \subseteq \mathbb{C}$, what is $\int_\gamma 1$?

Remark: The length of a curve $\gamma \subseteq \mathbb{C}$ parametrized by $p\colon [a,b] \to \mathbb{C}$, denoted $\operatorname{len}(\gamma)$ is defined as

$$\operatorname{len}(\gamma) = \int_a^b |p'(t)|\mathrm{d}t.$$

Contour integral

Definition

Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p\colon [a,b] \to \mathbb{C}$, let $f : \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

$$\int_\gamma f := \int_a^b f(p(t))p'(t)\mathrm{d}t.$$

Fact: The value of the integral does not depend on the choice of the parametrization of γ; it only depends on γ itself, including its orientation.

Example: For a curve $\gamma \subseteq \mathbb{C}$, what is $\int_\gamma 1$?

Remark: The length of a curve $\gamma \subseteq \mathbb{C}$ parametrized by $p\colon [a,b] \to \mathbb{C}$, denoted $\mathrm{len}(\gamma)$ is defined as

$$\mathrm{len}(\gamma) = \int_a^b |p'(t)|\mathrm{d}t.$$

Properties of $\int_\gamma$:

- Let $-\gamma$ denote the curve obtained from γ by reversing its orientation. Then $\int_{-\gamma} f = -\int_\gamma f$.

Contour integral

Definition

Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p\colon [a,b] \to \mathbb{C}$, let $f\colon \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

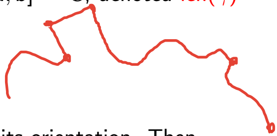
$$\int_\gamma f := \int_a^b f(p(t))p'(t)\mathrm{d}t.$$

Fact: The value of the integral does not depend on the choice of the parametrization of γ; it only depends on γ itself, including its orientation.

Example: For a curve $\gamma \subseteq \mathbb{C}$, what is $\int_\gamma 1$?

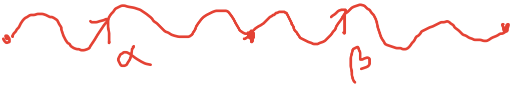
Remark: The length of a curve $\gamma \subseteq \mathbb{C}$ parametrized by $p\colon [a,b] \to \mathbb{C}$, denoted $\mathrm{len}(\gamma)$ is defined as

$$\mathrm{len}(\gamma) = \int_a^b |p'(t)|\mathrm{d}t.$$



Properties of $\int_\gamma$:

- Let $-\gamma$ denote the curve obtained from γ by reversing its orientation. Then $\int_{-\gamma} f = -\int_\gamma f$.
- If γ is the concatenation of two curves α and β, then $\int_\gamma f = \int_\alpha f + \int_\beta f$.



Contour integral

Definition

Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p \colon [a, b] \to \mathbb{C}$, let $f : \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

$$\int_\gamma f := \int_a^b f(p(t))p'(t)\mathrm{d}t.$$

Fact: The value of the integral does not depend on the choice of the parametrization of γ; it only depends on γ itself, including its orientation.

Example: For a curve $\gamma \subseteq \mathbb{C}$, what is $\int_\gamma 1$?

Remark: The length of a curve $\gamma \subseteq \mathbb{C}$ parametrized by $p \colon [a, b] \to \mathbb{C}$, denoted $\mathrm{len}(\gamma)$ is defined as

$$\mathrm{len}(\gamma) = \int_a^b |p'(t)|\mathrm{d}t.$$

Properties of $\int_\gamma$:

- Let $-\gamma$ denote the curve obtained from γ by reversing its orientation. Then $\int_{-\gamma} f = -\int_\gamma f$.
- If γ is the concatenation of two curves α and β, then $\int_\gamma f = \int_\alpha f + \int_\beta f$.
- For a parametrization p of γ, we have the estimate

$$\left|\int_\gamma f\right| = \left|\int_a^b f(p(t))p'(t)\mathrm{d}t\right| \le \int_a^b |f(p(t))| \cdot |p'(t)|\mathrm{d}t \le \sup_{z\in\gamma} |f(z)| \cdot \mathrm{len}(\gamma).$$

Primitive function

Definition

Let f be a function on a domain $\Omega \subseteq \mathbb{C}$. A function $F\colon \Omega \to \mathbb{C}$ is a primitive function (or antiderivative) of f on Ω, if for every $z \in \Omega$, we have $F'(z) = f(z)$.

In particular, if $\gamma \subseteq \Omega$ is a closed curve, then $\int_\gamma f = 0$.

Primitive function

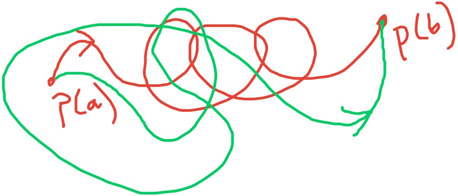
Definition

Let f be a function on a domain $\Omega \subseteq \mathbb{C}$. A function $F\colon \Omega \to \mathbb{C}$ is a primitive function (or antiderivative) of f on Ω, if for every $z \in \Omega$, we have $F'(z) = f(z)$.

Observation

If f has an antiderivative F on Ω, and $\gamma \subseteq \Omega$ is a curve parametrized by $p\colon [a,b] \to \Omega$, then

$$\int_\gamma f = \int_a^b f(p(t))p'(t)dt = \int_a^b F(p(t))'dt = F(p(b)) - F(p(a)).$$



In particular, if $\gamma \subseteq \Omega$ is a closed curve, then $\int_\gamma f = 0$.

Primitive function

Definition

Let f be a function on a domain $\Omega \subseteq \mathbb{C}$. A function $F\colon \Omega \to \mathbb{C}$ is a primitive function (or antiderivative) of f on Ω, if for every $z \in \Omega$, we have $F'(z) = f(z)$.

Observation

If f has an antiderivative F on Ω, and $\gamma \subseteq \Omega$ is a curve parametrized by $p\colon [a,b] \to \Omega$, then

$$\int_\gamma f = \int_a^b f(p(t))p'(t)dt = \int_a^b F(p(t))'dt = F(p(b)) - F(p(a)).$$

Corollary

Let F is an antiderivative of f on a domain Ω, and let $u, v \in \Omega$ be two points. Then for any curve $\gamma \subseteq \Omega$ connecting u with v and oriented from u to v, we have

$$\int_\gamma f = F(v) - F(u).$$

Primitive function

Definition

Let f be a function on a domain $\Omega \subseteq \mathbb{C}$. A function $F\colon \Omega \to \mathbb{C}$ is a primitive function (or antiderivative) of f on Ω, if for every $z \in \Omega$, we have $F'(z) = f(z)$.

Observation

If f has an antiderivative F on Ω, and $\gamma \subseteq \Omega$ is a curve parametrized by $p\colon [a, b] \to \Omega$, then

$$\int_\gamma f = \int_a^b f(p(t))p'(t)dt = \int_a^b F(p(t))'dt = F(p(b)) - F(p(a)).$$

Corollary

Let F is an antiderivative of f on a domain Ω, and let $u, v \in \Omega$ be two points. Then for any curve $\gamma \subseteq \Omega$ connecting u with v and oriented from u to v, we have

$$\int_\gamma f = F(v) - F(u).$$

In particular, if $\gamma \subseteq \Omega$ is a closed curve, then $\int_\gamma f = 0$.

Local antiderivatives of analyic functions

t

Fact

Let f be analytic in z_0, with an expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ of radius of convergence ρ. Then the function $F\colon \mathcal{N}_{<\rho}(z_0) \to \mathbb{C}$ defined by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} = (a_{-1}) + \frac{a_0}{1}(z-z_0) + \frac{a_1}{2}(z-z_0)^2 + \vdots$$

is an antiderivative of f on $\mathcal{N}_{<\rho}(z_0)$.

Fact

Let f be analytic in z_0, with an expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ of radius of convergence ρ. Then the function $F\colon \mathcal{N}_{<\rho}(z_0) \to \mathbb{C}$ defined by

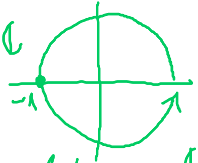
$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

is an antiderivative of f on $\mathcal{N}_{<\rho}(z_0)$.

Example: Let $k \in \mathbb{Z}$, let γ be the (counterclockwise) unit circle, parametrized by $p(t) = \exp(it)$ with $t \in [-\pi, \pi]$. What is $\int_\gamma z^k$?

$k \geq 0$: z^k is analytic on $\mathbb{C}$

z^k has antiderivative $\frac{z^{k+1}}{k+1}$ on $\mathbb{C}$



$\Rightarrow \int_\gamma z^k = 0$

$k \leq -2$: analytic on $\mathbb{C} \setminus \{0\}$, $\frac{z^{k+1}}{k+1}$ analytic on $\mathbb{C} \setminus \{0\}$

$\Rightarrow \int_\gamma z^k = 0$

$k=-1$: $\int_\gamma \frac{1}{z} = \int_{-\pi}^{\pi} \frac{1}{e^{it}} \cdot i \cdot e^{it} \, dt = 2\pi i$

Local antiderivatives of analyic functions

Fact

Let f be analytic in z_0, with an expansion $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ of radius of convergence ρ. Then the function $F\colon \mathcal{N}_{<\rho}(z_0) \to \mathbb{C}$ defined by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$$

is an antiderivative of f on $\mathcal{N}_{<\rho}(z_0)$.

Example: Let $k \in \mathbb{Z}$, let γ be the (counterclockwise) unit circle, parametrized by $p(t) = \exp(it)$ with $t \in [-\pi, \pi]$. What is $\int_\gamma z^k$?

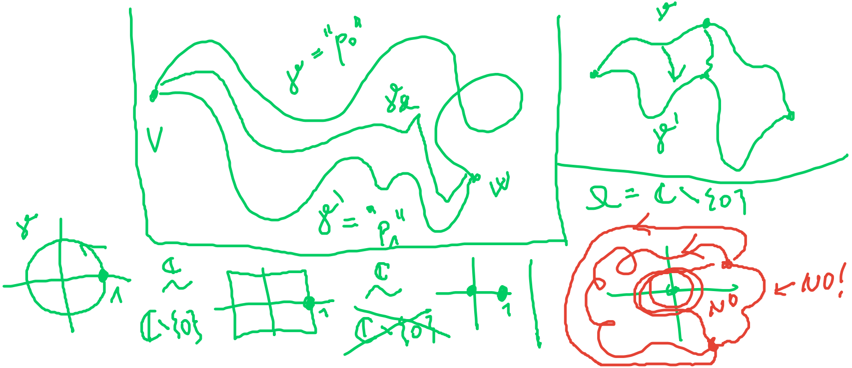
Note: Since $\int_\gamma \frac{1}{z} = 2\pi i \neq 0$, it follows that $f(z) = 1/z$ has no antiderivative on any domain containing γ, even though it is analytic on the domain $\mathbb{C} \setminus \{0\}$.

Curve homotopy

Definition

Let $\cancel{\omega}\ \Omega \subseteq \mathbb{C}$ be a domain, let γ and γ' be two curves in Ω, both starting in the same point v and ending in the same point w. We say that γ and γ' are fixed-endpoint homotopic (or just homotopic) in Ω if there is a continuous function $\Gamma(t, q) \colon [0, 1] \times [0, 1] \to \Omega$ with the following properties:

- For every $q \in [0, 1]$, the function $p_q \colon [0, 1] \to \Omega$ defined as $p_q(t) = \Gamma(t, q)$ is a parametrization of a curve γ_q starting in v and ending in w.
- The function $p_0(t)$ is a parametrization of γ, and the function $p_1(t)$ is a parametrization of γ'.



Curve homotopy

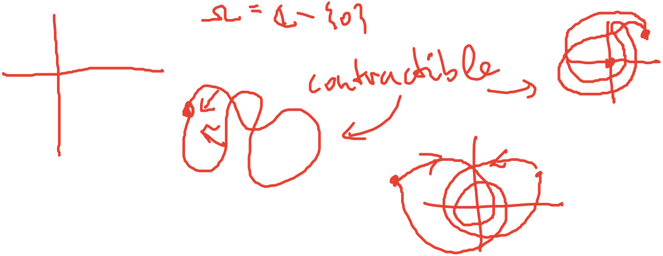
Definition

Let $\omega \subseteq \mathbb{C}$ be a domain, let γ and γ' be two curves in Ω, both starting in the same point v and ending in the same point w. We say that γ and γ' are fixed-endpoint homotopic (or just homotopic) in Ω if there is a continuous function $\Gamma(t,q)\colon [0,1]\times[0,1] \to \Omega$ with the following properties:

- For every $q \in [0,1]$, the function $p_q\colon [0,1] \to \Omega$ defined as $p_q(t) = \Gamma(t,q)$ is a parametrization of a curve γ_q starting in v and ending in w.
- The function $p_0(t)$ is a parametrization of γ, and the function $p_1(t)$ is a parametrization of γ'.

Definition

A closed curve γ is contractible (in Ω) if it is homotopic to a single point.



Curve homotopy

Definition

Let $\omega \subseteq \mathbb{C}$ be a domain, let γ and γ' be two curves in Ω, both starting in the same point v and ending in the same point w. We say that γ and γ' are fixed-endpoint homotopic (or just homotopic) in Ω if there is a continuous function $\Gamma(t, q)\colon [0, 1] \times [0, 1] \to \Omega$ with the following properties:

- For every $q \in [0, 1]$, the function $p_q\colon [0, 1] \to \Omega$ defined as $p_q(t) = \Gamma(t, q)$ is a parametrization of a curve γ_q starting in v and ending in w.
- The function $p_0(t)$ is a parametrization of γ, and the function $p_1(t)$ is a parametrization of γ'.

Definition

A closed curve γ is contractible (in Ω) if it is homotopic to a single point.

Fact

If γ is a simple closed curve inside a domain Ω such that $\mathrm{Int}(\gamma) \subseteq \Omega$*, then γ is contractible in Ω.*

Invariance of integral under homotopy

Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

$$\int_{\gamma} f = \int_{\gamma'} f.$$

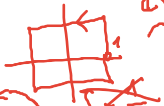
Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

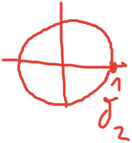
$$\int_\gamma f = \int_{\gamma'} f.$$

In particular, if a closed curve γ is contractible in Ω, then $\int_\gamma f = 0$.

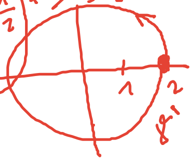
$\int_\gamma \frac{1}{z} = 2\pi i$, $\gamma =$



$\mathbb{C}\setminus\{0\}$ $\sim$

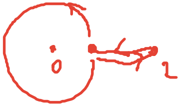


$\| \int_{\gamma'} \frac{1}{z} = \int_\gamma \frac{1}{z} = \cancel{\int_{\gamma_1} \frac{1}{z}} + \int_{\gamma_2} \frac{1}{z} + \cancel{\int_{\gamma_3} \frac{1}{z}}$



$\gamma_1 = -\gamma_3$

γ



$\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \sim \gamma'$

concatenation

Invariance of integral under homotopy

Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

$$\int_{\gamma} f = \int_{\gamma'} f.$$

In particular, if a closed curve γ is contractible in Ω, then $\int_{\gamma} f = 0$.

Proof idea: Let $\Gamma(t, q)\colon [0,1] \times [0,1] \to \Omega$ be a function witnessing the homotopy of γ and γ'.

Invariance of integral under homotopy

Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

$$I(0) = \int_{\gamma} f = \int_{\gamma'} f. = I(1)$$

In particular, if a closed curve γ is contractible in Ω, then $\int_{\gamma} f = 0$.

Proof idea: Let $\Gamma(t,q) \colon [0,1] \times [0,1] \to \Omega$ be a function witnessing the homotopy of γ and γ'.

For $q \in [0,1]$, let γ_q be the curve parametrized by $p_q(t) = \Gamma(t,q)$, and let $I(q) := \int_{\gamma_q} f$. We claim that $I(q)$ is a constant function of q on $[0,1]$.

$\gamma_0 = \gamma$, $\gamma_1 = \gamma'$

Invariance of integral under homotopy

Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

$$\int_\gamma f = \int_{\gamma'} f.$$

In particular, if a closed curve γ is contractible in Ω, then $\int_\gamma f = 0$.

Proof idea: Let $\Gamma(t, q) \colon [0, 1] \times [0, 1] \to \Omega$ be a function witnessing the homotopy of γ and γ'.

For $q \in [0, 1]$, let γ_q be the curve parametrized by $p_q(t) = \Gamma(t, q)$, and let $I(q) := \int_{\gamma_q} f$. We claim that $I(q)$ is a constant function of q on $[0, 1]$.

To see this, we choose $q \in [0, 1]$ and show that $I(q)$ is constant on a neighborhood of q.

Invariance of integral under homotopy

Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

$$\int_{\gamma} f = \int_{\gamma'} f.$$

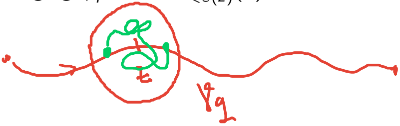
In particular, if a closed curve γ is contractible in Ω, then $\int_{\gamma} f = 0$.

Proof idea: Let $\Gamma(t, q)\colon [0, 1] \times [0, 1] \to \Omega$ be a function witnessing the homotopy of γ and γ'.

For $q \in [0, 1]$, let γ_q be the curve parametrized by $p_q(t) = \Gamma(t, q)$, and let $I(q) := \int_{\gamma_q} f$. We claim that $I(q)$ is a constant function of q on $[0, 1]$.

To see this, we choose $q \in [0, 1]$ and show that $I(q)$ is constant on a neighborhood of q.

The function f is analytic in every point $z \in \gamma_q$, hence there is an $\varepsilon(z) > 0$ such that f is analytic on $\mathcal{N}_{<\varepsilon(z)}(z)$ and therefore f has a primitive function F_z on $\mathcal{N}_{<\varepsilon(z)}(z)$. In particular, changing γ_q inside $\mathcal{N}_{<\varepsilon(z)}(z)$ does not affect the value $I(q)$.



Invariance of integral under homotopy

Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

$$\int_\gamma f = \int_{\gamma'} f.$$

In particular, if a closed curve γ is contractible in Ω, then $\int_\gamma f = 0$.

Proof idea: Let $\Gamma(t, q)\colon [0,1] \times [0,1] \to \Omega$ be a function witnessing the homotopy of γ and γ'.

For $q \in [0,1]$, let γ_q be the curve parametrized by $p_q(t) = \Gamma(t, q)$, and let $I(q) := \int_{\gamma_q} f$. We claim that $I(q)$ is a constant function of q on $[0,1]$.

To see this, we choose $q \in [0,1]$ and show that $I(q)$ is constant on a neighborhood of q.

The function f is analytic in every point $z \in \gamma_q$, hence there is an $\varepsilon(z) > 0$ such that f is analytic on $\mathcal{N}_{<\varepsilon(z)}(z)$ and therefore f has a primitive function F_z on $\mathcal{N}_{<\varepsilon(z)}(z)$. In particular, changing γ_q inside $\mathcal{N}_{<\varepsilon(z)}(z)$ does not affect the value $I(q)$.

By compactness of γ_q, there is a finite set $P \subseteq \gamma_q$ such that $\gamma_q \subseteq \bigcup_{z \in P} \mathcal{N}_{<\varepsilon(z)}(z)$.

$S \subseteq \mathbb{C}$ is compact $\Leftrightarrow$ S is closed and bounded

S is compact $\overset{\text{def}}{\Leftrightarrow}$ for any collection of open sets covering S there is a finite subcollection covering S.

Invariance of integral under homotopy

Fact

If f is analytic on a domain Ω, and if γ and γ' are homotopic in Ω, then

$$\int_\gamma f = \int_{\gamma'} f.$$

In particular, if a closed curve γ is contractible in Ω, then $\int_\gamma f = 0$.

Proof idea: Let $\Gamma(t,q)\colon [0,1]\times[0,1] \to \Omega$ be a function witnessing the homotopy of γ and γ'.

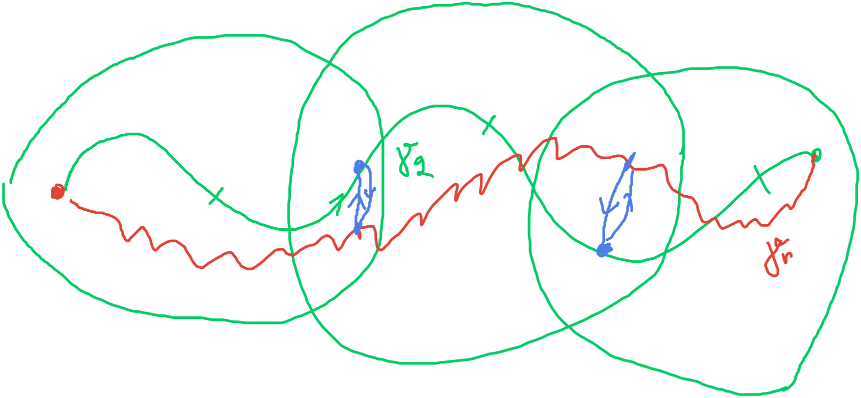
For $q \in [0,1]$, let γ_q be the curve parametrized by $p_q(t) = \Gamma(t,q)$, and let $I(q) := \int_{\gamma_q} f$. We claim that $I(q)$ is a constant function of q on $[0,1]$.

To see this, we choose $q \in [0,1]$ and show that $I(q)$ is constant on a neighborhood of q.

The function f is analytic in every point $z \in \gamma_q$, hence there is an $\varepsilon(z) > 0$ such that f is analytic on $\mathcal{N}_{<\varepsilon(z)}(z)$ and therefore f has a primitive function F_z on $\mathcal{N}_{<\varepsilon(z)}(z)$. In particular, changing γ_q inside $\mathcal{N}_{<\varepsilon(z)}(z)$ does not affect the value $I(q)$.

By compactness of γ_q, there is a finite set $P \subseteq \gamma_q$ such that $\gamma_q \subseteq \bigcup_{z\in P} \mathcal{N}_{<\varepsilon(z)}(z)$.

For r "close enough" to q, the curve γ_r is also inside $\bigcup_{z\in P} \mathcal{N}_{<\varepsilon(z)}(z)$, and we can modify γ_q into γ_r by operations that preserve the value of the integral, hence $I(q) = I(r)$ for r close enough to q. (See picture on next slide.)



$$\int_{\gamma_r} f = \int_{\gamma_q} f$$