$\nearrow L(B) = Y \cdot \Lambda(B)$ ... Cox P

9. Let Φ be a mixed Poisson point process with the driving measure $Y \cdot \Lambda$, where Y is a non-negative random variable and Λ is a locally finite diffuse measure. Determine the covariance $\mathrm{cov}(\Phi(B_1), \Phi(B_2))$ for $B_1, B_2 \in \mathcal{B}_0$ and show that it is non-negative.

$\to \Phi(B_1 \setminus B_2) + \Phi(B_1 \cap B_2)$

we know: $\mathrm{cov}(\Phi(B_1), \Phi(B_2)) = \ldots \quad (*)$

$\hookrightarrow \Phi(B_2 \setminus B_1) + \Phi(B_1 \cap B_2)$

$\mathrm{var}\, \Phi(B) = ?$

$$\mathbb{E}[\Phi(B)] = \mathbb{E}[\mathbb{E}[\underbrace{\Phi(B) \mid Y}_{Po(Y \cdot \Lambda(B))}]] = \mathbb{E}[Y \Lambda(B)] = \Lambda(B) \cdot \mathbb{E}Y$$

$$\mathbb{E}\Phi(B)^2 = \mathbb{E}[\mathbb{E}[\Phi(B)^2 \mid Y]] = \ldots$$

$\nearrow$ assume: $\mathbb{E}Y^2 < \infty$

$$\mathrm{var}\, \Phi(B) = \underbrace{\Lambda(B)^2 \cdot \mathrm{var}\, Y}_{\text{variance of cond. expectation}} + \underbrace{\Lambda(B) \cdot \mathbb{E}Y}_{\text{expectation of conditional variance}}$$

$$\mathrm{cov}(\Phi(B_1), \Phi(B_2)) = \mathrm{cov}(\Phi(B_1 \setminus B_2), \Phi(B_1 \cap B_2)) + \mathrm{cov}(\Phi(B_1 \cap B_2), \Phi(B_2 \setminus B_1)) + \mathrm{cov}(\Phi(B_1 \setminus B_2), \Phi(B$$
$$+ \mathrm{var}\, \Phi(B_1 \cap B_2)$$

assume $A, B \in \mathcal{B}$ disjoint:

$$\mathrm{cov}(\Phi(A), \Phi(B)) = \mathbb{E}\Phi(A)\Phi(B) - \overbrace{(\mathbb{E}\Phi(A))(\mathbb{E}\Phi(B))}^{(\mathbb{E}Y)^2 \Lambda(A)\Lambda(B)}$$

$$\mathbb{E}\Phi(A)\Phi(B) = \mathbb{E}[\mathbb{E}[\Phi(A)\Phi(B) \mid Y]] = \mathbb{E}[\mathbb{E}[\Phi(A) \mid Y] \cdot \mathbb{E}[\Phi(B) \mid Y]] =$$

independent under the condition, $\Phi(A) \sim Po($

$$= \mathbb{E}[Y \Lambda(A) \cdot Y \Lambda(B)] = \Lambda(A)\Lambda(B)\, \mathbb{E}Y^2$$

$$\mathrm{cov}(\Phi(A), \Phi(B)) = \Lambda(A)\Lambda(B) \cdot \mathrm{var}\, Y$$

$$\mathrm{cov}(\Phi(B_1), \Phi(B_2)) = \mathrm{var}\, Y \, [\overbrace{\Lambda(B_1 \setminus B_2) \cdot \Lambda(B_2 \setminus B_1) + \Lambda(B_1 \setminus B_2)\Lambda(B_1 \cap B_2)}^{\Lambda(B_1 \setminus B_2) \cdot \Lambda(B_2)} +$$
$$+ \underbrace{\Lambda(B_2 \setminus B_1) \cdot \Lambda(B_1 \cap B_2) + \Lambda(B_1 \cap B_2)\Lambda(B_1 \cap B_2)}_{\Lambda(B_1 \cap B_2)\Lambda(B_2)}] + \Lambda(B_1 \cap B_2) \cdot \mathbb{E}Y =$$

$=$

$$= \underbrace{\operatorname{var} Y}_{\geq 0} \, [\underbrace{\Lambda(B_1)\Lambda(B_2)}_{\geq 0}] + \underbrace{\mathbb{E} Y}_{\geq 0} \cdot \underbrace{\Lambda(B_1 \cap B_2)}_{\geq 0} \geq 0$$

makes sense: $\Phi(B_1)$ is high $\Rightarrow$ $\Phi(B_2)$ is likely to be high too because Y is likely to be high.
(even if B_1, B_2 disjoint).