Limitní věty pro martingaly

Definice: (Posloupnost martingalových diferencí)

Buď $\{X_{n,j},\ n \geq 1,\ j = 1, 2, \dots n_j\}$ systém náhodných veličin na $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_{n,j},\ n \geq 1,\ j = 1, 2, \dots n_j\}$ systém σ-algeber

(i) $\mathcal{F}_{n,j-1} \subset \mathcal{F}_{n,j} \subset \mathcal{F}$ $\qquad X_{n,j}$ je $\mathcal{F}_{n,j}$ měřitelné $\forall n, j$.

(ii) $E X_{n,j}^2 = \sigma_{n,j}^2 < \infty \quad \forall n, j$

(iii) $E[X_{n,j} \mid \mathcal{F}_{n,j-1}] = 0$ s.j. $\quad \forall n, j$

Pak $\{X_{n,j}\}$ nazveme posloupností martingalových diferencí (pmd) (vůči $\{\mathcal{F}_{n,j}\}$)

Veta 1: (McLeishova)

Buď $\{X_{n,j}\}$ je pmd $n \geq 1$, $j = 1, \dots, r_n$,

(1) $\sup_n E\left(\max_{j \leq r_n} X_{n,j}^2\right) < \infty$

(2) $\max_{j \leq r_n} |X_{n,j}| \xrightarrow[n\to\infty]{P} 0$

(3) $\sum_{j=1}^{r_n} X_{n,j}^2 \xrightarrow[n\to\infty]{P} 1$

nechť $r_n \xrightarrow[n\to\infty]{} \infty$. Pak $S_n = \sum_{j=1}^{r_n} X_{n,j} \xrightarrow{d} N(0,1)$

poznamenujmo: $S_{n,k} = \sum_{j=1}^{k} X_{n,j}$ $\quad$ $S_{n,k}$ je $\mathcal{F}_{n,k}$-merljiva sl. veličina

$$j<k \qquad E\left[S_{n,k} \mid \mathcal{F}_{n,j}\right] = E\Big[\underbrace{S_{n,j}}_{\mathcal{F}_{n,j}\text{ merlj.}} + \sum_{l=j+1}^{k} X_{n,l} \,\Big|\, \mathcal{F}_{n,j}\Big] = S_{n,j} + \sum_{l=j+1}^{k} \underbrace{E\left[X_{n,l} \mid \mathcal{F}_{n,j}\right]}_{\stackrel{s.g.}{=} 0}$$

$$\stackrel{s.g.}{=} S_{n,j}$$

$\{S_{n,k},\ k=0,1,\dots,r_n\}$ je diskretni L_2-martingal

Důkaz: $Z_{n,j} = X_{n,j} \cdot \mathbb{1}\left[\sum_{k=1}^{j-1} X_{n,k}^2 \leq 2\right]$

$$\sum_{j=1}^{r_n} Z_{n,j}^2 \leq 2 + X_{n,J}^2 \qquad J = \min\left\{ j : \sum_{k=1}^{j} X_{n,k}^2 > 2 \right\}$$

$$\left(= r_n \text{ pokud } \sum_{k=1}^{r_n} X_{n,k}^2 \leq 2 \right)$$

$Z_{n,j}$ ~~tvoří~~ pmd ($Z_{n,j}$ je $\mathcal{F}_{n,j}$ měřitelná n.v., $E Z_{n,j}^2 < \infty$

$$E\left[Z_{n,j} \mid \mathcal{F}_{n,j-1} \right] = E\left[X_{n,j} \cdot \mathbb{1}\left[\sum_{k=1}^{j-1} X_{n,k}^2 \leq 2\right] \,\Big|\, \underline{\mathcal{F}_{n,j-1}} \right] = \mathbb{1}(\quad) \cdot E\left[X_{n,j} \mid \mathcal{F}_{n,j-1} \right]$$

$$= 0$$

$$P\left[\max_j |Z_{n,j} - X_{n,j}| > 0\right] = P\left[\exists j \sum_{l=1}^{j-1} X_{n,l}^2 > 2\right]$$

$$\leq P\left[\sum_{l=1}^{r_n} X_{n,l}^2 > 2\right] \xrightarrow[n\to\infty]{P} 0$$

$$P\left[\left|\sum_{j=1}^{r_n} Z_{n,j} - \sum_{j=1}^{r_n} X_{n,j}\right| > 0\right] \xrightarrow[n\to\infty]{P} 0$$

Pokud dokážeme $\sum_{j=1}^{r_n} Z_{n,j} \xrightarrow[n\to\infty]{d} N(0,1)$, pak z Cramérovy-Sluckého

věty plyne i $\sum_{j=1}^{r_n} X_{n,j} \xrightarrow[n\to\infty]{d} N(0,1)$

Ukážeme, že $E \exp\left(it \sum_{j=1}^{n} Z_{n,j}\right) \xrightarrow{n\to\infty} e^{-t^2/2} \quad \forall t \in \mathbb{R}$

$$\boxed{\log(1+ix) = ix - \frac{(ix)^2}{2} - r(x)}$$

$$r(x) = x^3\left(i\underbrace{\left(\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \dots\right)}_{x=1 \quad \approx 0{,}21} + x\underbrace{\left(\frac{1}{4} - \frac{x^2}{6} + \frac{x^4}{8} - \dots\right)}_{0{,}15}\right)$$

$|r(x)| \le |x|^3$ pro $|x| < 1$

$$ix = \log(1+ix) - \frac{x^2}{2} + r(x)$$

$$e^{ix} = (1+ix) \cdot e^{-x^2/2} \cdot e^{r(x)}$$

$$\exp\left(it \sum_{j=1}^{n} Z_{n,j}\right) = \prod_{j=1}^{n} \exp(it Z_{n,j}) = \prod_{j=1}^{n} (1 + it Z_{n,j}) \exp\left(-\frac{t^2}{2} \sum_{j=1}^{n} Z_{n,j}^2 + \sum_{j=1}^{n} r(Z_{n,j})\right)$$

$$= \underbrace{\prod_{j=1}^{r_n} (1 + it Z_{n,j})\, e^{-t^2/2}}_{T_n} + \underbrace{\prod_{j=1}^{r_n} (1 + it Z_{n,j}) \left[\exp\left\{ -\frac{t^2}{2} \sum_{j=1}^{r_n} Z_{nj}^2 + \sum_{j=1}^{r_n} r(t Z_{n,j}) \right\} - e^{-t^2/2} \right]}_{V_n}$$

$$E \qquad ET_n \overset{?}{\longrightarrow} 1 \qquad EV_n \overset{?}{\longrightarrow} 0$$

$$E|T_n|^2 = E \prod_{j=1}^{r_n} (1 + t^2 Z_{n,j}^2) \leq E \prod_{j=1}^{J-1} e^{t^2 Z_{n,j}^2} (1 + t^2 Z_{nJ}^2) \cdot \underbrace{1 \cdot 1 \cdots 1}$$

$1 + x \leq e^x$

pro indexy do $J-1$ $\quad \sum_{\ell=1}^{J-1} X_{n,j}^2 \leq 2 < \sum_{\ell=1}^{J} X_{n,j}^2$

$Z_{nJ+1} = 0$ atd. ...

$$\leq 2$$

$$= E e^{t^2 \sum_{j=1}^{J-1} Z_{nj}^2} \cdot (1 + t^2 Z_{nJ}^2) \leq e^{2t^2} \cdot E(1 + t^2 X_{nJ}^2)$$

$$\sup_n E \max_{j \leq r_n} X_{nj}^2 < \infty$$

$E|T_n|^2$ je stejnoměrně omezená

$\sup_n E|T_n|^2 < \infty$ $\qquad$ $|T_n|$ je stejnoměrně integrovatelná posloupnost

$$\lim_{N\to\infty} \sup_n E|T_n| \cdot 1[|T_n| > N] = 0$$