Analytic combinatorics
Lecture 7

April 21, 2021

Meromorphic functions

Definition

Let Ω be a domain. A function f is meromorphic on Ω if for every p of Ω, f is either analytic in p or has a pole in p.

Suppose p is a zero of order1 d of h, i.e., $h(z) = \sum_{n \geq d} a_n(z-p)^n$ on some $\mathcal{N}_{<\varepsilon}(p)$, with $a_d \neq 0$.

It follows that $\frac{h(z)}{(z-p)^d} = \sum_{n \geq 0} a_{n+d}(z-p)^n$ defines an analytic function h^* on $\mathcal{N}_{<\varepsilon}(p)$, with $h^*(p) = a_d \neq 0$.

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A function f is meromorphic in a domain Ω iff there are two functions g and h analytic on Ω, with h not identically zero on Ω, such that

$$f(z) = \frac{g(z)}{h(z)}$$

for every $z \in \Omega \setminus \{z;\ h(z) = 0\}$.

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Approximating meromorphic functions by rational functions

Proposition

Let f be meromorphic on a domain Ω, and suppose it has only finitely many poles in Ω. Then there is a rational function $R(z)$ such that the function $g(z) = f(z) - R(z)$ has an analytic continuation to Ω. Moreover, the only poles of R are the poles of f.

Suppose now that Ω contains $\mathcal{N}_{\leq\rho}(0)$ for some $\rho > 0$. Then

- *the radius of convergence of $g(z)$ around 0 is greater than ρ,*
- *and therefore its exponential growth rate is smaller than $\frac{1}{\rho}$,*
- *hence $[z^n]g(z) \leq \frac{1}{\rho^n}$ for n large enough, and*
- *most importantly, $[z^n]f(z) = [z^n]R(z) + O(\frac{1}{\rho^n})$.*

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Corollary

Let Ω, f, g and R be as above, and suppose f is analytic in 0. Let $[z^n]f(z)$ denote the coefficient of degree n in the power series expansion of f in 0. In particular, we know that

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If $z_0 \in \Omega \setminus P$, then clearly g is analytic in z_0.

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If $z_0 = p \in P$, then on a punctured neighborhood of p we have

$$g(z) = f(z) - R_p(z) - \sum_{q \in P \setminus \{p\}} R_q(z) = \left(\sum_{n \geq 0} a_n (z-p)^n \right) - \sum_{q \in P \setminus \{p\}} R_q(z),$$

which is analytic in $z_0 = p$. □

Recall: Ordered set partitions of $[n]$ are the ordered sequences of nonempty disjoint sets whose union is $[n]$. Let p_n be their number. Their EGF is $f(z) = \frac{1}{2-\exp(z)}$. $= \sum_{n\geq 0} p_n \frac{z^n}{n!}$

Hence, $\frac{p_n}{n!}$ has radius of convergence $\ln 2$ and exponential growth rate $\frac{1}{\ln 2}$.

Goal: Find a better estimate for p_n.

$f(z) = \frac{1}{2-e^z}$, meromorphic on $\mathbb{C}$

Poles of f: $P = \{\ln 2, \dots\} = \{z \in \mathbb{C};\ e^z = 2\}$

$z = x + iy$, $x, y \in \mathbb{R}$, $e^z = e^{x+iy} = e^x \cdot e^{iy} =$

$= e^x(\cos y + i \sin y)$. $|e^z| = |e^x \cdot e^{iy}| = |e^x| = 2$

$\Rightarrow x = \ln 2$

$e^z = 2 \Rightarrow \operatorname{Im}(e^z) = 0 = e^x \cdot \sin y \Rightarrow y \in \{k\pi, k \in \mathbb{Z}\}$

$\operatorname{Re}(e^z) = 2 = e^x \cdot \cos y \Rightarrow \cos y = 1 \Rightarrow y \in \{2k\pi, k \in \mathbb{Z}\}$

$P = \{\ln 2 + i 2k\pi;\ k \in \mathbb{Z}\}$

$f(z) = \frac{1}{2-e^z}$; $p_k := \ln 2 + 2k\pi i$, $k \in \mathbb{Z}$

Choose $\rho \in (|p_0|, |p_1|)$

$\Omega = \mathcal{N}_{<\rho}(0)$

f has pole p_0 in Ω

$f = R + g$ ← analytic in Ω

↑ rational, with pole p_0

$f(z) = \frac{-\frac{1}{2}}{z-p_0} + a_0 + a_1(z-p_0) + \dots$

$R(z) = \frac{-\frac{1}{2}}{z-p_0} = \frac{1}{2\cdot\ln 2} \cdot \frac{1}{1-\frac{z}{\ln 2}} = \dots$

$= \frac{1}{2\cdot\ln 2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{\ln 2}\right)^n$

$[z^n] f(z) = [z^n] R + O\left(\frac{1}{\rho^n}\right)$ for any $\rho < |p_1|$

$\frac{p_n}{n!} = \frac{1}{2\cdot\ln 2} \cdot \left(\frac{1}{\ln 2}\right)^n$

What is the order of p_0 as pole of f? $= \min_{d\in\mathbb{N}} : (z-p_0)^d f(z)$ is analytic

$(z-p_0) f(z) = \frac{z-p_0}{2-e^z}$ is bounded around p_0, for $z \in \mathbb{R}$, hence it has no pole, hence it has a removable singularity: $\lim_{z\to p_0} \frac{z-p_0}{2-e^z} = -\frac{1}{2}$

↑ l'Hospital

Example: What is the probability that a random permutation of $[n]$ has no cycle of length k? (Assume k fixed, $n \to \infty$.)

$k=1 \dashrightarrow \frac{1}{e}$

$k \in \mathbb{N}$ fixed: $g_n :=$ # of perms of $[n]$ without cycles of length k, wanted: $\frac{g_n}{n!}$

$f(z) :=$ EGF of perms without k-cycles

$g(z)$: EGF of perms with 1 cycle, whose length $\neq k$

$$g(z) = \sum_{\substack{n=1 \\ n \neq k}}^{\infty} (n-1)! \frac{z^n}{n!} = \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \right) - \frac{z^k}{k}, \quad \frac{g^2(z)}{2} = \text{EGF of perms with two cycles, of length} \neq k$$

$$f(z) = 1 + g(z) + \frac{g^2(z)}{2!} + \dots = \exp(g(z)) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \right) \cdot \exp\left(-\frac{z^k}{k} \right)$$

(EGF of perms)

$$f(z) = \left(\sum_{n=0}^{\infty} n! \frac{z^n}{n!} \right) \cdot e^{-z^k/k} = \frac{e^{-z^k/k}}{1-z}$$

$$f(z) = \frac{\exp\left(-\frac{z^k}{k}\right)}{1-z}, \text{ meromorphic on } \mathbb{C}$$

$$\frac{g_n}{n!} \xrightarrow{n\to\infty} e^{-1/k}$$

only pole: $p=1$, $\Omega := \mathbb{C}$

$f(z)(z-1)$ is analytic in $p=1$ => ~~degree~~ order of $p=1$

$$f(z) = \frac{a_{-1}}{z-p} + a_0 + a_1(z-p) + a_2(z-p)^2 + \dots \quad (p=1)$$

$$a_{-1} = \lim_{z\to p} (z-p)\cdot f(z) = -\exp\left(-\frac{1}{k}\right)$$

$$f(z) = \frac{-\exp\left(-\frac{1}{k}\right)}{z-1} + (\text{something analytic in } \mathbb{C})$$

$\forall \varepsilon > 0$

$$\frac{g_n}{n!} = [z^n] f(z) = [z^n] e^{-1/k} \cdot \left(\sum_{n=0}^{\infty} z^n\right) + - O(\varepsilon^n)$$

Complex integration

What we know:

- For a real function $f : [a, b] \to \mathbb{R}$, we are familiar with the notion of integral $\int_a^b f(t)\mathrm{d}t = \int_a^b f$.

The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a, b)$ and $p(a) = p(b)$.

The bounded one is the interior of γ, *denoted* $\mathrm{Int}(\gamma)$, *the other is the exterior of* γ, *denoted* $\mathrm{Ext}(\gamma)$.

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- This can be extended to continuous complex-valued functions $f\colon [a,b] \to \mathbb{C}$ by $\int_a^b f = \int_a^b \Re(f) + i\Im(f) = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition

Let $[a,b]$ be a real interval with $a < b$, let $p\colon [a,b] \to \mathbb{C}$ be a continuous function with a finite derivative $p'(t)$ everywhere on (a,b) except at most finitely many points, and with finite right (and left) derivative everywhere on $[a,b)$ (or $(a,b]$, respectively). The curve parametrized by p is the set $\gamma = \{p(t);\ t \in [a,b]\}$, together with the orientation from $p(a)$ to $p(b)$. The function p is then the parametrization of γ.
The curve is said to be...

- simple if p is injective,
- closed if $p(a) = p(b)$,
- simple closed if p is injective on $[a,b)$ and $p(a) = p(b)$.

The bounded one is the interior of γ, denoted $\mathrm{Int}(\gamma)$*, the other is the exterior of γ, denoted* $\mathrm{Ext}(\gamma)$.

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Contour integral

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Let $\gamma \subseteq \mathbb{C}$ be a curve with parametrization $p\colon [a,b] \to \mathbb{C}$, let $f : \gamma \to \mathbb{C}$ be a function. The contour integral of f along γ, denoted $\int_\gamma f$ is defined as

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- Let $-\gamma$ denote the curve obtained from γ by reversing its orientation. Then $\int_{-\gamma} f = -\int_\gamma f$.
- If γ is the concatenation of two curves α and β, then $\int_\gamma f = \int_\alpha f + \int_\beta f$.

Primitive function

Definition

Let f be a function on a domain $\Omega \subseteq \mathbb{C}$. A function $F\colon \Omega \to \mathbb{C}$ is a primitive function (or antiderivative) of f on Ω, if for every $z \in \Omega$, we have $F'(z) = f(z)$.

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Observation

If f has a primitive function F on Ω, and $\gamma \subseteq \Omega$ is a curve parametrized by $p \colon [a, b] \to \Omega$, then

$$\int_\gamma f = \int_a^b f(p(t))p'(t)\mathrm{d}t = \int_a^b F(p(t))'\mathrm{d}t = F(p(b)) - F(p(a)).$$

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Example: Let $k \in \mathbb{Z}$, let γ be the counterclockwise unit circle, parametrized by $p(t) = \exp(it)$ with $t \in [-\pi, \pi]$. What is $\int_\gamma z^k$?