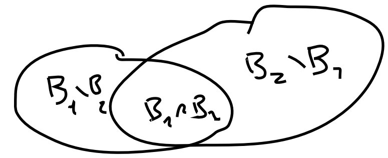
5. Let Φ be a Poisson point process with the intensity measure Λ. Determine the covariance $\mathrm{cov}(\Phi(B_1), \Phi(B_2))$ for $B_1, B_2 \in \mathcal{B}$.

B_1, B_2 may not be disjoint!



$\Phi(B_1) \sim Po(\Lambda(B_1))$

$\Phi(B_2) \sim Po(\Lambda(B_2))$

$\Phi(B_1) = \Phi(B_1 \setminus B_2) + \Phi(B_1 \cap B_2$

$$\mathrm{cov}(\Phi(B_1), \Phi(B_2)) = \begin{cases} \xrightarrow{B_1 \cap B_2 = \emptyset} 0 \\ \xrightarrow{B_1 \cap B_2 \neq \emptyset} = \dots \end{cases}$$

$$\dots = \mathrm{cov}\Big(\Phi(B_1 \setminus B_2) + \Phi(B_1 \cap B_2), \Phi(B_2 \setminus B_1) + \Phi(B_1 \cap B_2)\Big) =$$

$$= \mathrm{cov}\big(\Phi(B_1 \setminus B_2), \Phi(B_2 \setminus B_1)\big) + \mathrm{cov}\big(\Phi(B_1 \setminus B_2), \Phi(B_1 \cap B_2)\big) +$$

$$+ \mathrm{cov}\big(\Phi(B_1 \cap B_2), \Phi(B_2 \setminus B_1)\big) + \mathrm{cov}\big(\Phi(B_1 \cap B_2), \Phi(B_1 \cap B_2)\big) =$$

$$= 0 + 0 + 0 + \mathrm{var}\, \underbrace{\Phi(B_1 \cap B_2)}_{\sim Po(\Lambda(B_1 \cap B_2))} = \Lambda(B_1 \cap B_2).$$