# NMFM 402, Practical 6, GLM - confidence intervals

Ex 1. a) Recall the approximate (asymptotic) result from general MLE theory:

$$\underset{\substack{\uparrow \\ \text{MLE estimator} \\ \text{(random vector)}}}{\beta_{MLE}} \overset{d}{\approx} N\Big(\underset{\substack{\uparrow \\ \text{true} \\ \text{value}}}{\beta}, \underset{\substack{\uparrow \\ \text{inverse of the} \\ \text{Fisher information matrix}}}{\mathbb{J}^{-1}}\Big)$$

We can further approximate the Fisher info. matrix:

$$\mathbb{J} = \mathbb{J}(\beta) \approx \mathbb{J}(\underset{\substack{\uparrow \\ \text{point estimate} \\ \text{from the data} \\ \text{(realization of } \beta_{MLE})}}{\hat{\beta}})$$

For Poisson model with log link ~~function~~ function:
(see Practical 4, Ex 1 a))

$$\mathbb{J}(\hat{\beta}) = Z^T \mathbb{D} \cdot Z, \quad \text{where}$$

$Z$ ... design matrix

$\mathbb{D} = \text{diag}\{d_m\}$, with $d_m = v_m \cdot \exp\{(Z\hat{\beta})_m\}$

(**NOTE**: For general Tweedie models, we have

$$d_m = \frac{v_m}{\varphi \cdot \mu_m^{p-2}}$$ )

Denote $\mathbb{C} = \left[J(\hat{\beta})\right]^{-1}$ ... approximate covariance matrix for $\beta_{MLE}$.

Tor, we have $\beta_{MLE} \overset{d}{\approx} N(\beta, \mathbb{C})$, and

$$\beta_{MLE, e} \overset{d}{\approx} N(\beta_e, c_{ee})$$

The approximate confidence interval for $\beta_e$ is

$$\mathbb{P}\left[\beta_e \in \underline{\left(\hat{\beta}_e - u_{1-\frac{\alpha}{2}} \sqrt{c_{ee}} ; \hat{\beta}_e + u_{1-\frac{\alpha}{2}} \sqrt{c_{ee}}\right)}\right] \approx 1-\alpha,$$

~~while~~ where $u_{1-\frac{\alpha}{2}}$ is $1-\frac{\alpha}{2}$ quantile of $N(0,1)$.

b) Using the fact that $f(x) = e^x$ is a monotone (~~inst~~ increasing) function, we can construct approximate confidence interval for $\exp(\beta_e)$ directly from a):

$$\mathbb{P}\left[\exp(\beta_e) \in \underline{\underline{\left(\exp(\hat{\beta}_e - u_{1-\frac{\alpha}{2}} \sqrt{c_{ee}}) ; \exp(\hat{\beta}_e + u_{1-\frac{\alpha}{2}} \sqrt{c_{ee}})\right)}}\right] \approx 1-\alpha$$

c) Log link function implies $\mathbb{E} Y_n = \exp((Z\beta)_n)$.

From previous considerations:

$$Z\beta_{MLE} \overset{d}{\approx} N(Z\beta ; Z \cdot C \cdot Z^T)$$

$$(Z\beta_{MLE})_n \overset{d}{\approx} N\left((Z\beta)_n ; (Z C Z^T)_{n,n}\right)$$

Similarly to b) we get the confidence intervals

$$\mathbb{P}\left[\mathbb{E} Y_n \in \underline{\left(\exp(a_n), \exp(b_n)\right)}\right] \approx 1-\alpha \quad , \text{ where}$$

$$a_n = (\mathbb{Z}\hat{\beta})_n - u_{1-\frac{\alpha}{2}} \cdot \sqrt{(\mathbb{Z} C \mathbb{Z}^T)_{n,n}}$$

$$b_n = (\mathbb{Z}\hat{\beta})_n + u_{1-\frac{\alpha}{2}} \sqrt{(\mathbb{Z} C \mathbb{Z}^T)_{n,n}} .$$

<u>NOTE</u>: The knowledge of (approximate) joint distribution of $\beta_{MLE}$ (resp. $\mathbb{Z}\beta_{MLE}$) makes it possible to calculate (approximate) confidence regions for $\beta$ (resp. $\mathbb{E}\mathbb{Y} = \mathbb{Z}\beta$).