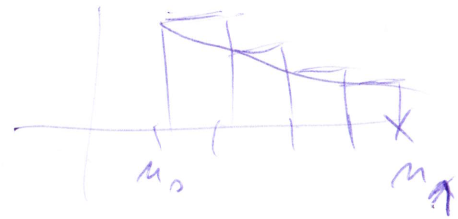
$\boxed{\text{Věta L7.19}}$ (integrální kritérium konvergence řad)

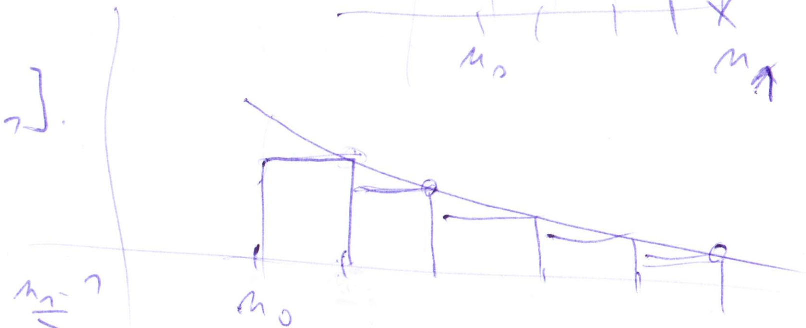
Nechť f je nezáporná, nerostoucí a spojitá na intervalu $[n_0-1, \infty)$ pro nějaké $n_0 \in \mathbb{N}$. Nechť pro posloupnost a_n platí $a_n = f(n)$ pro všechna $n \geq n_0$. Pak

$$(N)\int_{n_0}^{\infty} f(x)\,dx < \infty \iff \sum_{n=1}^{\infty} a_n \text{ konverguje}$$



Dk: Nechť $n_1 \geq n_0$ a mějme

$D = \{n_0, n_0+1, \dots, n_1\}$ intervalu $[n_0, n_1]$.



Funkce f je nerostoucí, a tedy

$$S(f,D) = a_{n_0} + a_{n_0+1} + \dots + a_{n_1-1} = \sum_{i=n_0}^{n_1-1} a_i$$

$$s(f,D) = a_{n_0+1} + a_{n_0+2} + \dots + a_{n_1} = \sum_{i=n_0+1}^{n_1} a_i .$$

Protože f je spojitá na $[n_0, n_1]$, platí

$$\underline{\sum_{i=n_0+1}^{n_1} a_i} = s(f,D) \leq (R)\int_{n_0}^{n_1} f(x)\,dx = \underline{(N)\int_{n_0}^{n_1} f(x)\,dx} \leq S(f,D) = \underline{\sum_{i=n_0}^{n_1-1} a_i} . \quad (*)$$

nechť $\int_{n_0}^{\infty} f(x)\,dx$ konverguje. Pak je

$F(x) = \int_{n_0}^{x} f(t)\,dt$, $t \in [n_0, \infty)$ je primitivní k $f(x)$ na (n_0, ∞). (V 7.9)

Tedy $\forall n_1 \geq n_0$ z ⊛

$$\int_{n_0}^{\infty} f(x)\,dx = \lim_{x\to\infty} F(x) - \overset{0}{F(n_0)} = \lim_{x\to\infty} \int_{n_0}^{x} f(t)\,dt \overset{\text{"}f\geq 0\text{"}}{\geq} \lim_{n\to\infty} \int_{n_0}^{n} f(t)\,dt = \left\{\text{polož } n_1 = [x]\right\}$$

$$\geq \lim_{n\to\infty} \sum_{i=n_0+1}^{n} a_i = \sum_{i=n_0+1}^{\infty} a_i. \Rightarrow \sum_{i=1}^{\infty} a_i \ K$$

neboť $a_i \geq 0$

Obráceně nechť $\sum_{i=1}^{\infty} a_i \ K \Rightarrow \sum_{i=n_0+1}^{\infty} a_i \ K$

z ⊛ $$\sum_{i=n_0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=n_0}^{n-1} a_i \geq \lim_{n\to\infty} \int_{n_0}^{n} f(t)\,dt =$$

$$= \lim_{n\to\infty} F(n) = \lim_{x\to\infty} F(x)$$

$F(x)$ je neklesající.

Tedy $\lim_{x\to\infty} F(x) \in \mathbb{R} \Rightarrow \int_{n_0}^{\infty} f(x)\,dx \ K$

□

<u>Příklad:</u> $$\sum_{i=1}^{n} i\cdot \log i \approx \int_{1}^{n} \underset{u'}{x}\cdot \underset{v}{\log x}\,dx = \left[\frac{x^2}{2}\cdot \log x\right]_1^n - \int_1^n \frac{x^2}{2}\cdot\frac{1}{x}\,dx =$$

$u = \frac{x^2}{2}$, $v' = \frac{1}{x}$

$$= \frac{n^2}{2}\cdot\log n - \frac{1}{2}\cdot\left[\frac{x^2}{2}\right]_1^n = \frac{n^2}{2}\log n - \frac{1}{4}n^2 + \frac{1}{4} \approx \frac{n^2}{2}\log n.$$

Příklad: Stirlingova formule $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$

$$b_n \overset{\text{TRIK}}{=} \log \frac{n!}{n^n} = \log n! - \log n^n = \sum_{k=1}^{n} \log k - n\cdot \log n$$

Abelova parciální sumace $\quad \sum a_n b_n = \sum A_k\cdot(b_k - b_{k+1})$

$$\sum_{k=1}^{n} \log k = \sum_{k=1}^{n} 1\cdot \log k = \sum_{k=1}^{n-1} k\cdot(\log k - \log(k+1)) + n\log n =$$

$$= \sum_{k=2}^{n} (k-1)(\log(k-1) - \log k) + n\log n =$$

$$= \sum_{k=2}^{n} (k-1) \log\left(1-\frac{1}{k}\right) + n\log n$$

Taylorův polynom $\quad f(t) = \log(1-t) \quad t\in\left(0,\tfrac{1}{2}\right]$

Lagrangeův tvar zbytku Taylorova polynomu

$$f(t) - T_2^{0,f}(t) = \frac{1}{3!}\cdot f^{(3)}(\xi(t))\cdot t^3 \qquad \xi(t)\in(0,t)\subset\left(0,\tfrac{1}{2}\right)$$

$$f'(t) = \frac{-1}{1-t}$$

$$f''(t) = \frac{-1}{(1-t)^2}$$

$$f'''(t) = \frac{-2}{(1-t)^3}$$

$$\log(1-t) = -t - \frac{1}{2}t^2 + \frac{1}{3!}\cdot\frac{-2}{(1-\xi(t))^3}\cdot t^3$$

$t = \frac{1}{k},\ k\geq 2 \qquad 0 \leq \theta_k \leq \frac{8}{3} \leq 3$

$$\log\left(1-\frac{1}{k}\right) = -\frac{1}{k} - \frac{1}{2}\frac{1}{k^2} - \frac{1}{3}\frac{1}{(1-\xi(\frac{1}{k}))^3}\cdot\frac{1}{k^3} = -\frac{1}{k} - \frac{1}{2k^2} - \frac{\theta_k}{k^3}$$

$$b_n = \sum_{k=2}^{n} (k-1)\cdot \log\left(1-\frac{1}{k}\right) = \sum_{k=2}^{n}(k-1)\cdot\left(-\frac{1}{k} - \frac{1}{2k^2} - \frac{\vartheta_k}{k^3}\right) = \qquad \underline{17-4} \quad 0 \le \vartheta_k \le 3$$

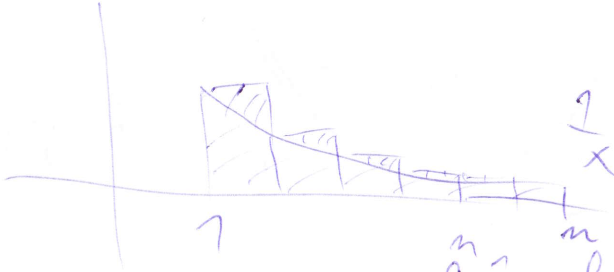
$$= \sum_{k=2}^{n}\left(-1 - \frac{1}{2k} - \frac{\vartheta_k}{k^2} + \frac{1}{k} + \frac{1}{2k^2} + \frac{\vartheta_k}{k^3}\right) =$$

$$= \sum_{k=2}^{n}\left(-1 + \frac{1}{2k} + \underbrace{\frac{1}{k^2}\cdot\left(\frac{1}{2} + \frac{\vartheta_k - \vartheta_k\cdot k}{k}\right)}_{|\quad| \le 4}\right) =$$

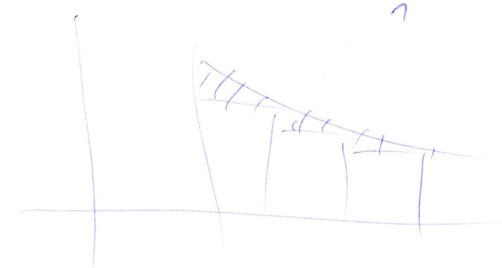
$$= -(n-2) + \frac{1}{2}\cdot\sum_{k=2}^{n}\frac{1}{k} + \underbrace{\sum_{k=2}^{\infty}\frac{\psi_k}{k^2}}_{\in \mathbb{R}} \qquad |\psi_k| \le 4$$

- $\exists \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log n\right] \in \mathbb{R}$

$$\sum_{k=1}^{n-1}\frac{1}{k} - \log n \nearrow \qquad \sum_{k=2}^{n}\frac{1}{k} - \log n \searrow$$



$$\int_1^n \frac{1}{x} = \log n$$



~~Zatím~~ Odtud plyne

$$\exists \lim_{n\to\infty}\left(b_n + n - \frac{1}{2}\log n\right) \in \mathbb{R}$$

$$\|$$

$$\lim_{n\to\infty}\left(\log\frac{n!}{n^n} + \log e^n - \log\sqrt{n}\right) = \lim_{n\to\infty}\log\left(\frac{n!}{\left(\frac{n}{e}\right)^n\cdot\sqrt{n}}\right)$$

$$\implies \exists \lim_{n\to\infty}\frac{n!}{\left(\frac{n}{e}\right)^n\cdot\sqrt{n}} = a \in \mathbb{R}\setminus\{0\}$$

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \cdot \sqrt{n}} = a$$

Příklad Wallisova formule

$$I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx = \int_0^{\frac{\pi}{2}} \underbrace{\sin x}_{v'} \cdot \underbrace{(\sin x)^{n-1}}_{u} \, dx =$$

$v = -\cos x$

$$= \left[-\cos x \cdot (\sin x)^{n-1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) \cdot (n-1) \cdot (\sin x)^{n-2} \cdot \cos x \, dx =$$

$$= (n-1) \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)(\sin x)^{n-2} \, dx = (n-1)\, I_{n-2} - (n-1) \cdot I_n$$

$$\Rightarrow \boxed{I_n = \frac{n-1}{n} \cdot I_{n-2}} \quad \& \quad I_0 = \frac{\pi}{2} \quad \& \quad I_1 = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

Zřejmě

$$I_{2n+1} = \int_0^{\frac{\pi}{2}} (\sin^{2n+1} x) \leq I_{2n} = \int_0^{\frac{\pi}{2}} (\sin x)^{2n} \leq I_{2n-1} = \int_0^{\frac{\pi}{2}} (\sin x)^{2n-1}$$

$$\frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots 1 \leq \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{\pi}{2} \leq \frac{2n-2}{2n-1} \cdots 1$$

$$\frac{(2n)!!}{(2n+1)!!} \leq \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \leq \frac{(2n-2)!!}{(2n-1)!!} \qquad \Big/ \frac{(2n)!!}{(2n-1)!!}$$

$$\frac{1}{2n+1} \cdot \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \leq \frac{\pi}{2} \leq \frac{1}{2n} \left[\frac{(2n)!!}{(2n-1)!!}\right]^2$$

$$\left(a_n \leq \frac{\pi}{2} \leq \frac{2n+1}{2n} a_n\right)$$

Víme $\lim_{n\to\infty} \frac{2n}{2n+1} = 1 \overset{\text{Zmáčk}}{\Longrightarrow} \lim_{n\to\infty} \frac{1}{2n+1} \cdot \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 = \frac{\pi}{2}$

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \cdot \sqrt{n}} = a \quad \Big| \quad \lim_{n\to\infty} \frac{1}{2n+1} \cdot \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 = \frac{\pi}{2}$$

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$$/ \cdot \frac{(2n)!!}{(2n)!!}$$

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{1}{2n+1} \cdot \left[\frac{((2n)!!)^2}{(2n)!}\right]^2 =$$

$$\left\{ (2n)!! = 2n \cdot 2\cdot(n-1) \cdot 2\cdot(n-2) \cdots 2 = 2^n \cdot n! \right\}$$

$$= \lim_{n\to\infty} \frac{1}{2n+1} \cdot \left[\frac{(2^n \cdot n!)^2}{(2n)!}\right]^2 =$$

$$= \lim_{n\to\infty} \left(\frac{n!}{\left(\frac{n}{e}\right)^n \cdot \sqrt{n}}\right)^4 \cdot \left(\frac{\left(\frac{2n}{e}\right)^{2n} \cdot \sqrt{2n}}{(2n)!}\right)^2 \cdot \frac{1}{2n+1} \cdot \left[\frac{\left(2^n \cdot \left(\frac{n}{e}\right)^n \cdot \sqrt{n}\right)^2}{\left(\frac{2n}{e}\right)^{2n} \cdot \sqrt{2n}}\right]^2$$

$$= a^4 \cdot \frac{1}{a^2} \cdot \lim_{n\to\infty} \frac{1}{2n+1} \left[\frac{\cancel{2^{2n}} \cdot \frac{\cancel{n^{2n}}}{\cancel{e^{2n}}} \cdot n}{\frac{\cancel{2^{2n}} \cdot \cancel{n^{2n}}}{\cancel{e^{2n}}} \cdot \sqrt{2n}}\right]^2 =$$

$$= a^2 \cdot \lim_{n\to\infty} \frac{n^2}{(2n+1) \cdot 2n} = \frac{a^2}{4} \Rightarrow a = \sqrt{2\pi}$$