**Left and right isotopes.** Let $Q$ be a loop, and let $e$ be an element of $Q$. A full name for the loop $(Q,*)$, $x*y = x/e \cdot y$, might be the *left loop principal isotope* induced by $e$. For simplicity let this be called a *left isotope.* Similarly, $x*y = x \cdot f\backslash y$ defines the *right isotope* induced by $f$.

Suppose that $x*y = x \cdot f\backslash y$. What are the left isotopes of $(Q,*)$? Denote by $/\!\!/$ the right division in $(Q,*)$. Thus $x/\!\!/y = z \Leftrightarrow x = z*y \Leftrightarrow x = z \cdot f\backslash y$ $\Leftrightarrow z = x/(f\backslash y)$. The operation of the left isotope of $(Q,*)$ induced by $e$ thus is $x/\!\!/e*y = (x/(f\backslash e))\cdot(f\backslash y)$. If $(f,e)$ runs through $Q\times Q$, then $(f\backslash e, e)$ runs through $Q\times Q$ too. This implies:

> *(1) The set of left isotopes of right isotopes of $Q$ coincides with the set of all principal loop isotopes of $Q$, and*
> *(2) the set of right isotopes of left isotopes of $Q$ also coincides with the set of all principal loop isotopes of $Q$.*

The statement above was proved under the assumption that $Q$ is a loop. In fact it holds for every quasigroup $Q$.

**LIP loops.** A loop $Q$ is said to possess left inverses if

$$\forall x \in Q\ \exists y \in Q \text{ such that } L_y = L_x^{-1}.$$

As will be proved, if $Q$ possesses left inverses then

$$x(1/x \cdot y) = y,\ 1/x \cdot (xy) = y,\ x\backslash 1 \cdot (xy) = y \text{ and } x(x\backslash 1 \cdot y) = y$$

for all $x, y \in Q$. On the other hand, if any of these identities holds, then $Q$ possesses left inverses. To prove the latter is easy since $x(1/x \cdot y) = y$ means that $L_x L_{1/x} = \mathrm{id}_Q$, and the other identities may be intepreted similarly.

Let $x, y \in Q$ be such that $L_x^{-1} = L_y$. Then $y(xz) = z$ for all $z \in Q$. Setting $z = 1$ yields $yx = 1$ and $y = 1/x$. Setting $z = x\backslash 1$ yields $y = x\backslash 1$. The assumption $L_x^{-1} = L_y$ also means that $x(yz) = z$ for all $z \in Q$. Thus $xy = 1$, and $y = x\backslash 1$. Setting $z = y\backslash 1$ gives $x = y\backslash 1$. Hence $y = 1/(y\backslash 1) = 1/x$ too.

A loop that possesses left inverses thus fulfils all of the four identities. Therefore $1/x = x\backslash 1$ for each $x \in Q$. If $1/x = x\backslash 1$, then the notation $x^{-1}$ may be used.

Saying that $Q$ 'possesses left inverses' refers to the fact that the set $\{L_x;\, x \in Q\}$ is closed under the taking of an inverse permutation. A more traditional way of saying that $Q$ possesses left inverses is to say that $Q$ has the *left inverse property* (LIP). Furthermore, instead of saying that $Q$ has the left inverse property it is usual to say that $Q$ is a *LIP loop.* As explained above, if $Q$ is a LIP loop, then

$$\forall x, y \in Q \quad x \cdot x^{-1}y = x^{-1} \cdot xy = y.$$

This may be also expressed as $L_x^{-1} = L_{x^{-1}}$. RIP loops fulfil $yx \cdot x^{-1} = y = yx^{-1} \cdot x$. That means $R_x^{-1} = R_{x^{-1}}$.

**Left isotopes and LIP loops.** Let $(Q,*)$ be a left isotope of a loop $Q$, say $x*y = x/e \cdot y$. For $x \in Q$ denote by $\lambda_x$ the left translation of $(Q,*)$, and by $L_x$ the left translation of $Q$. Then $\lambda_x = L_{x/e}$. Hence

$$\{\lambda_x;\ x \in Q\} = \{L_x;\ x \in Q\}.$$

This implies that $Q$ is a LIP loop (i.e., posseses left inverses) if and only if $(Q,*)$ is a LIP loop. We have proved:

> (1) A left isotope of a LIP loop is a LIP loop; and
> (2) A right isotope of a RIP loop is a RIP loop.

**Left Bol loops.** Let $Q$ be a loop. The following is equivalent:

(1) The set $L_Q = \{L_x;\ x \in Q\}$ is closed under *twists* (i.e., if $\alpha, \beta \in L_Q$, then $\alpha\beta\alpha \in L_Q$);
(2) the set $L_Q = \{L_x;\ x \in Q\}$ is closed under *inverted twists* (i.e., if $\alpha, \beta \in L_Q$, then $\alpha\beta^{-1}\alpha \in L_Q$);
(3) if $x, y \in Q$, then $L_x L_y L_x = L_{x\cdot yx}$;
(4) each right isotope of $Q$ is a LIP loop;
(5) each isotope of $Q$ is a LIP loop;
(6) $Q$ satisfies the identitity $x(y \cdot xz) = (x \cdot yx)z$.

*Proof.* First note that $L_x L_y L_x = L_{x\cdot yx}$ means that $x \cdot (y \cdot xz) = (x \cdot yx)z$. Hence (3) $\Leftrightarrow$ (6). If $L_x L_y L_x = L_z$, then $z = L_z(1) = L_x L_y L_x(1) = x \cdot yx$. Hence (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (6).

If $Q$ satisfies (2) then $Q$ is a LIP loop since $L_x^{-1} = L_1 L_x^{-1} L_1 \in L_Q$. Thus $L_x = L_{x^{-1}}^{-1}$ and $L_x L_y L_x = L_x L_{y^{-1}}^{-1} L_x \in L_Q$, for any $x, y \in Q$. Hence (2) $\Rightarrow$ (1). To prove the converse by the same method it suffices to show that the identity of (6) implies the left inverse property. That follows from setting $y = 1/x$. Indeed, then $x(1/x \cdot xz) = xz$, and so $1/x \cdot xz = z$. Therefore (1) $\Rightarrow$ (2). We have shown that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (6).

Clearly, (5) $\Rightarrow$ (4). The converse follows from the fact that each loop isotope of $Q$ is isomorphic to a principal loop isotope, each principal loop isotope is a left isotope of a right isotope, and each left isotope of a LIP loop is a LIP loop.

To finish it thus suffices to verify (2) $\Leftrightarrow$ (4). Consider $f \in Q$ and denote by $\lambda_x$ the left translation of $(Q, *)$, $x * y = x \cdot f\backslash y$. Clearly, $\lambda_x = L_x L_f^{-1}$. What does it mean that the set $\{L_x L_f^{-1};\ x \in Q\}$ is closed under inversions? This means that for each $x \in Q$ there exists $y \in Q$ such that $L_x L_f^{-1} L_y L_f^{-1} = \mathrm{id}_Q$. Hence $L_f^{-1} L_x L_f^{-1} L_y = \mathrm{id}_Q$, $L_y^{-1} L_f L_x^{-1} L_f = \mathrm{id}_Q$ and $L_y = L_f L_x^{-1} L_f$. In other words, $(Q, *)$ is a LIP loop if and only if for each $x \in Q$ there exists $y \in Q$ such that $L_f L_x^{-1} L_f = L_y$. This is true for all $f \in Q$ if and only if the set $L_Q$ is closed under inverted twists. $\square$

The identity $x(y \cdot xz) = (x \cdot yx)z$ is known as the *left Bol law.* Loops that fulfil this law are called *left Bol loops* of just *Bol loops.* The *right Bol* loops are those that fulfil the *right Bol law* $z(xy \cdot x) = (zx \cdot y)x$.

**Moufang loops.** A loop $Q$ is called *Moufang* if it is both the left and the right Bol loop. Moufang loops are thus those loops that satisfy both identities $x(y\cdot xz) = (x \cdot yx)z$ and $z(xy \cdot x) = (zx \cdot y)x$.

The variety of Moufang loops is much bigger than the variety of groups. Nevertheless, Moufang loops are not so far from groups as other loop varieties. This is well documented by the *Moufang's theorem*:

> Let $Q$ be a Moufang loop. If $x, y, z \in Q$ are such that $x \cdot yz = xy \cdot z$, then $\langle x, y, z\rangle$ is a group.

The theorem of Moufang may be rephrased by saying that associating elements generate an associating subloop (i.e., a group).

The proof of the theorem is relatively complicated and needs several pages.

**Operations in a LIP loop.** The left division of a LIP loop is dispensible since $x\backslash y = x^{-1}y$ for all elements $x$ and $y$ of a LIP loop $Q$. LIP loops may thus be considered as algebras in signature $(\cdot, /, {}^{-1}, 1)$ such that

$$x \cdot 1 = x = 1 \cdot x,\ (x^{-1})^{-1} = x,\ x^{-1} \cdot xy = y \text{ and } (y/x)x = y = (yx)/x.$$

**IP loops.** A loop $Q$ is said to have the *invere property* if it is both a LIP loop and a RIP loop. Loops with inverse property are called IP loops. An IP loop may be considered as an algebra in signature $(\cdot, {}^{-1}, 1)$ such that

$$x \cdot 1 = x = 1 \cdot x,\ (x^{-1})^{-1} = x \text{ and } x^{-1} \cdot xy = y = yx \cdot x^{-1}.$$

*Lemma.* If $x, y \in Q$ and $Q$ is an IP loop, then

$$(xy)^{-1} = y^{-1}x^{-1}.$$

*Proof.* The statement may be modified to $y^{-1} = (x\backslash y)^{-1}x^{-1}$, by writing $y$ as $x\backslash y$. Now,

$$\begin{aligned} y^{-1} = (x\backslash y)^{-1}x^{-1} \ &\Leftrightarrow\ y^{-1}x = (x\backslash y)^{-1} \ \Leftrightarrow\ x = y(x\backslash y)^{-1} \\ &\Leftrightarrow\ x \cdot (x\backslash y) = y \ \Leftrightarrow\ y = y. \end{aligned}$$

□

**IP Bol loops are Moufang.** A left Bol loop $Q$ is a LIP loop. A right Bol loop is a RIP loop. A Moufang loop is hence an IP loop. The statement to prove is:

*Lemma.* A RIP left Bol loop is Moufang.

*Proof.* In a left Bol loop $x(y \cdot xz) = (x \cdot yx)z$. If such loop is an IP loop, then

$$(x(y \cdot xz))^{-1} = (z^{-1}x^{-1} \cdot y^{-1})x^{-1} \text{ and } ((x \cdot yx)z)^{-1} \quad = z^{-1}(x^{-1}y^{-1} \cdot x^{-1}),$$

yielding thus the right Bol law. □

**Flexibility and the Moufang law.** The *flexible law* is the identity $x \cdot yx = xy \cdot x$. Note that a loop $Q$ is flexible if and only if $L_xR_x = R_xL_x$ for all $x \in Q$.

*Lemma.* A loop $Q$ is Moufang if and only if $Q$ is a flexible Bol loop.

*Proof.* Let $Q$ be a Moufang loop. Then $Q$ is an IP loop such that $x \cdot (y \cdot xz) = (x \cdot yx)z$ for all $x, y, z \in Q$. Setting $z = x^{-1}$ yields $xy = (x \cdot yx)x^{-1}$. Therefore $xy \cdot x = x \cdot yx$.

Let $Q$ be a left Bol loop that is flexible. It is enough to verify that $Q$ is a RIP loop. The flexibility induces the identity $x \cdot (y \cdot xz) = (xy \cdot x)z$. Setting $z = x^{-1}$ yields $xy = (xy \cdot x)x^{-1}$. □

**Two Moufang identities.** *Let $Q$ be a loop. The following is equivalent:*

(1) *$Q$ is Moufang;*
(2) *$Q$ fulfils $x(y \cdot xz) = (xy \cdot x)z$;*
(3) *$Q$ fulfils $z(x \cdot yx) = (zx \cdot y)x$;*
(4) *$(R_xL_x, L_x^{-1}, L_x) \in \mathrm{Atp}(Q)$ for all $x \in Q$; and*
(5) *$(R_x^{-1}, L_xR_x, R_x) \in \mathrm{Atp}(Q)$ for all $x \in Q$.*

*Proof.* Setting $z = 1$ yields the flexible law in both of the identities above. The flexible law changes them into a Bol identity. Flexible Bol loops are Moufang. It remains to observe that

$$x(y \cdot xz) = (xy \cdot x)z \ \Leftrightarrow\ L_x(yz) = x \cdot yz = (xy \cdot x)(x\backslash z) = R_xL_x(y) \cdot L_x^{-1}(z);$$

$$z(x \cdot yx) = (zx \cdot y)x \ \Leftrightarrow\ R_x^{-1}(z) \cdot L_xR_x(y) = (z/x)(x \cdot yx) = zy \cdot x = R_x(zy).$$

□

**Autotopisms describing Bol loops.** The left Bol loop identity may be expressed as $x \cdot yz = (x \cdot yx)(x\backslash z)$. Hence

$$Q \text{ is left Bol} \ \Leftrightarrow\ (L_xR_x, L_x^{-1}, L_x) \in \mathrm{Atp}(Q) \ \text{ for all } x \in Q;$$

$$Q \text{ is right Bol} \ \Leftrightarrow\ (R_x^{-1}, R_xL_x, R_x) \in \mathrm{Atp}(Q) \ \text{ for all } x \in Q.$$

**Switching translations.** Let $Q$ be an IP loop. Denote the operation of the inverse as a mapping $I$. Thus $I(x) = x^{-1}$ for each $x \in Q$. Then

$$IR_xI = L_x^{-1} \text{ and } IL_xI = R_x^{-1} \text{ for every } x \in Q.$$

*Proof.* If $x, y \in Q$, then $IR_xI(y) = I(y^{-1}x) = (y^{-1}x)^{-1} = x^{-1}y = L_x^{-1}(y)$. □

**Switching components of an isotopism.** *Suppose that $Q$ is an IP loop and that $\alpha, \beta, \gamma \in \mathrm{Sym}(Q)$. If $(\alpha, \beta, \gamma) \in \mathrm{Atp}(Q)$, then $(\gamma, I\beta I, \alpha) \in \mathrm{Atp}(Q)$.*

*Proof.* The assumption is that $\alpha(x)\beta(y) = \gamma(xy)$ for all $x, y \in Q$. This can be expressed as $\alpha(x) = \gamma(xy)(\beta(y))^{-1} = \gamma(xy) \cdot I\beta(y)$. Replacing $x$ with $xy^{-1}$ yields

$$\alpha(xI(y)) = \gamma(x) \cdot I\beta(y). \text{ Thus } \alpha(xy) = \gamma(x) \cdot I\beta I(y).$$

□

**The third Moufang identity.** *A loop $Q$ fulfils the identity*

$$xy \cdot zx = x(yz \cdot x) \;\Leftrightarrow\; (L_x, R_x, L_xR_x) \in \mathrm{Atp}(Q) \textit{ for each } x \in Q.$$

*Each such loop is a flexible IP loop.*

*Proof.* To get the RIP set $z = 1/x$. Then $xy = x((y \cdot 1/x)x)$, and thus $y = (y \cdot 1/x)x$ for all $x, y \in Q$. To get flexibility set $z = 1$. The flexibility implies that the identity is equivalent to its mirror image $xy \cdot zx = (x \cdot yz)x$. That yields the LIP. □

**The equivalence of Moufang identities.** *Let $Q$ be a loop. Each of the following identities is equivalent to $Q$ being Moufang:*

$$x(y \cdot xz) = (xy \cdot x)z, \tag{lM}$$

$$(zx \cdot y)x = z(x \cdot yx), \tag{rM}$$

$$xy \cdot zx = x(yz \cdot x), \textit{ and} \tag{mMl}$$

$$xy \cdot zx = (x \cdot yz)x. \tag{mMr}$$

*Proof.* We already know that (lM) = (rM). By flexibility, (mMl) = (mMr). Composing autotopism expressions of (lBol) and (rM) implies that

$$(L_xR_x, L_x^{-1}, L_x)\,(R_x^{-1}, L_xR_x, R_x) = (L_x, R_x, L_xR_x) \in \mathrm{Atp}(Q)$$

in every Moufang loop $Q$. Thus (lM) $\Rightarrow$ (mM). To get the converse implication note that switching components of $(L_x, R_x, L_xR_x)$ yields $(L_xR_x, IR_xI, L_x) = (R_xL_x, L_x^{-1}, L_x)$ since loops fulfilling (mM) are flexible IP loops. □

**Description of nuclei.** Similar technique may be used to prove that in a Moufang loop $N_\lambda(Q) = N_\rho(Q) = N_\mu(Q)$. Recall that if $Q$ is a loop, then

$$N_\lambda(Q) = \{a \in Q;\ a \cdot xy = ax \cdot y \text{ for all } x, y \in Q\};$$
$$N_\mu(Q) = \{a \in Q;\ x \cdot ay = xa \cdot y \text{ for all } x, y \in Q\}; \text{ and}$$
$$N_\rho(Q) = \{a \in Q;\ x \cdot ya = xy \cdot a \text{ for all } x, y \in Q\}.$$

It is clear that

$$a \in N_\lambda(Q) \;\Leftrightarrow\; (L_a, \mathrm{id}_Q, L_a) \in \mathrm{Atp}(Q), \text{ and}$$
$$a \in N_\rho(Q) \;\Leftrightarrow\; (\mathrm{id}_Q, R_a, R_a) \in \mathrm{Atp}(Q).$$

**Middle nucleus and translations.** Let $Q$ be a loop. Then

$$a \in N_\mu(Q) \;\Leftrightarrow\; (R_a, L_a^{-1}, \mathrm{id}_Q) \in \mathrm{Atp}(Q) \;\Leftrightarrow\; (R_a^{-1}, L_a, \mathrm{id}_Q) \in \mathrm{Atp}(Q).$$

*Proof.* Indeed, $x \cdot ay = xa \cdot y$ holds for all $x, y \in Q$ if and only if $xy = xa \cdot a\backslash y$ or, alternatively, $x/a \cdot ay = xy$, for all $x, y \in Q$. □

**Nuclei in Bol loops and Moufang loops.**

(1) Let $Q$ be left Bol. Then $N_\lambda(Q) = N_\mu(Q)$.
(2) Let $Q$ be right Bol. Then $N_\rho(Q) = N_\mu(Q)$.
(3) Let $Q$ be Moufang. Then $N_\lambda(Q) = N_\mu(Q) = N_\rho(Q)$.

*Proof.* It suffices to verify the first claim. Recall that in every left Bol loop $(L_x R_x, L_x^{-1}, L_x) \in \mathrm{Atp}(Q)$, for every $x \in Q$. The equality

$$(L_a^{-1}, \mathrm{id}_Q, L_a^{-1})\,(L_a R_a, L_a^{-1}, L_a) = (R_a, L_a^{-1}, \mathrm{id}_Q)$$

thus implies that $(L_a, \mathrm{id}_Q, L_a) \in \mathrm{Atp}(Q)$ if and only if $(R_a, L_a^{-1}, \mathrm{id}_Q) \in \mathrm{Atp}(Q)$. $\square$

**The left and right alternative laws.** These are the laws $x \cdot xz = xx \cdot z$ and $zx \cdot x = z \cdot xx$, respectively. Loops satisfying these laws are said to have the *left* or *right alternative property.* The loops themselves are then known as LAP and RAP loops.

The left Bol law $x(y \cdot xz) = (x \cdot yx)z$ yields the left alternative law by setting $y = 1$. Left Bol loops are thus LAP loops, while right Bol loops are RAP loops.

**Exercise.** A loop $Q$ is a left Bol loop if and only if each loop isotope of $Q$ fulfils the LAP. A loop $Q$ is a right Bol loop if and only if each loop isotope of $Q$ fulfils the RAP.

**Power associativity.** A loop $Q$ is said to be *left power associative* if it is LIP and fulfils

$$L_{x^i} L_{x^j}(y) = L_x^{i+j}(y) \text{ for all } x, y \in Q \text{ and all } i, j \in \mathbb{Z}.$$

The left power associativity may be paraphrased by saying that terms of the form

$$x^{\pm 1} x^{\pm 1} \cdots x^{\pm 1} y$$

are independent of bracketing.

It may be proved that left Bol loops are left power associative. The proof is not difficult.

**Diassociativity.** A loop $Q$ is said to be *diassociative* if the subloop $\langle x, y \rangle$ is associative (and thus a group) for any choice $x, y \in Q$.

Moufang loops are diassociative. This follows, e.g., from flexibility and Moufang's theorem. There exist direct proofs of diassociativity in Moufang loops. However, they are not much simpler than the proof of the Moufang's theorem.