Analytic combinatorics

Lecture 6

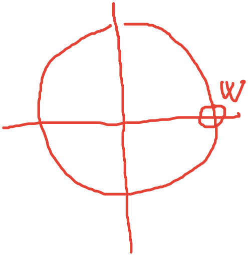
April 14, 2021

Prigsheim's theorem

Recall:

Fact (Pringsheim, Vivanti; 1890's)

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $\rho \in (0, +\infty)$*, and let us define* $f \colon \mathcal{N}_{<\rho}(0) \to \mathbb{C}$ *by* $f(z) = \sum_{n=0}^{\infty} a_n z^n$*. Then there is at least one point* w *with* $|w| = \rho$ *such that* f *has no analytic continuation to any domain containing* w*. If we additionally assume that* $a_n \geq 0$ *for all* n*, then the conclusion holds for* $w = \rho$*.*



Example: ordered set partitions

An ordered set partition of the set $[n]$ is an ordered sequence $(B_1, B_2, \ldots, B_k)$ of nonempty pairwise disjoint sets whose union is $[n]$. Let p_n be the number of ordered set partitions of $[n]$.

$p_0 = 1$

$p_1 = 1 \quad (\{1\})$

$p_2 = 3 \quad (\{1\}, \{2\}), \quad (\{2\}, \{1\}), \quad (\{1,2\})$

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Goal: find an estimate of p_n.

$$n\text{-th Bell number} \leq p_n \overset{?}{\leq} n!$$

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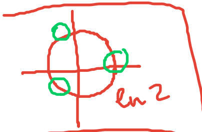
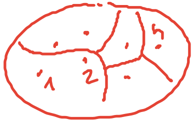
Pringsheim: $P(x)$ has radius of convergence $\ln 2$, i.e. $\left(\frac{1}{\ln 2}-\varepsilon\right)^n \overset{*}{\leq} \frac{p_n}{n!} \leq \left(\frac{1}{\ln 2}+\varepsilon\right)^n$

Goal: find an estimate of p_n.

Approach:

1. ✓ Find a generating function for ordered set partitions — EGF $P(x) = \sum_{n=0}^{\infty} p_n \frac{x^n}{n!}$
2. Apply Pringsheim's theorem





Partitions w. 1 block

$$B(x) = 1\cdot\frac{x^1}{1!} + 1\cdot\frac{x^2}{2!} + \dots + 1\cdot\frac{x^n}{n!} + \dots$$
$$= \exp(x) - 1$$

$B^2(x)$ is the EGF of ordered s.p. with 2 blocks

$k \in \mathbb{N}$: $B^k(x)$ … EGF of or. s. p. with k blocks

$$P(x) = 1 + B(x) + B^2(x) + \dots = \frac{1}{1-B(x)} = \frac{1}{2-\exp(x)}$$

$P(z) = \frac{1}{2-\exp(z)} : \Omega \to \mathbb{C}$, analytic on $\Omega := \{z \in \mathbb{C},\ \exp(z) \neq 2\}$

$Z := \{z \in \mathbb{C},\ \exp(z) = 2\}$

Rational functions

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Observe:

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- Suppose Q has k distinct roots. Let $r_1, \ldots, r_k$ be the distinct roots of Q, and let m_i be the multiplicity of the root r_i. Then $Q(z)$ can be written as $c \prod_{j=1}^{k} (z - r_j)^{m_j}$ for a constant $c \in \mathbb{C}$.

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- Suppose f is analytic in 0, and in particular $Q(0) \neq 0$. Then

$$Q(z) = Q(0) \prod_{j=1}^{k} \left(1 - \frac{z}{r_j}\right)^{m_j}.$$

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$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

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$$\left(\frac{1}{1-z}\right)^m = (1 + z + z^2 + \ldots)^m = \sum_{n=0}^{\infty} c_n z^n, \quad c_n = \#$$ of possibilities of writing n as a sum of m nonneg. integers

$$\left(\frac{1}{1-z}\right)^2 = (1 + z + z^2 + \ldots) \cdot (1 + z + z^2 + \ldots) = \sum_{n=0}^{\infty} (n+1) z^n$$

- Note:

$$\boxed{\frac{1}{\left(1 - \frac{z}{r}\right)^m} = \sum_{n=0}^{\infty} \underbrace{\binom{n+m-1}{m-1}}_{c_n} \frac{z^n}{r^n} \quad \text{for } z \in \mathcal{N}_{<r}(0).}$$

$$z \to \frac{z}{r}$$

Partial fraction decomposition

Fact (Partial fraction decomposition)

Suppose that $f(z) = \frac{P(z)}{Q(z)}$, *where* P *and* Q *are polynomials with no common roots,* $Q(0) \neq 0$, Q *has* k *distinct roots* $r_1, \ldots, r_k$, *the root* r_j *has multiplicity* m_j, *and* $|r_1| \leq |r_2| \leq \cdots \leq |r_k|$. *Then*

$$f(z) = R(z) + \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \frac{c_{j,\ell}}{\left(1 - \frac{z}{r_j}\right)^{\ell}},$$

$\in \mathbb{C}$

where $R(z)$ *is a polynomial of degree at most* $\deg(P) - \deg(Q)$, *and* $c_{j,\ell}$ *are constants.*

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In particular, for $\rho = |r_1| > 0$ *and any* $z \in \mathcal{N}_{<\rho}(0)$*, we have* $f(z) = \sum_{n=0}^{\infty} a_n z^n$*, where for every* $n > \deg(R)$*, we have*

$$a_n = \sum_{j=1}^{k} \frac{1}{r_j^n} \underbrace{\sum_{\ell=1}^{m_j} c_{j,\ell} \binom{n+\ell-1}{\ell-1}}_{\text{"some polynomial in } n \text{ of degree} \leq m_j - 1\text{"}}.$$

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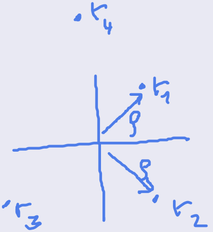

Consequently, if $\underbrace{\overset{\rho}{\overset{\shortparallel}{|r_1|}}}_{*} < |r_2|$, *then*

$$|a_n| = \frac{\Theta(n^{m_1 - 1})}{\rho^n}.$$

Pr.: $\left(\frac{1}{\rho} - \varepsilon\right)^n \overset{*}{\leq} |a_n| \leq \left(\frac{1}{\rho} + \varepsilon\right)^n$

Singularities, poles, zeros

Definition

Let Ω be a domain, let $f : \Omega \to \mathbb{C}$ a function analytic on Ω, let $p \in \mathbb{C}$ be a point in the complex plane. We say that p is an isolated singularity of f, if $\mathcal{N}^*_{<\varepsilon}(p) \subseteq \Omega$ for some $\varepsilon > 0$.

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We distinguish three types of isolated singularities:

- p is a removable singularity, if f has an analytic continuation to $\Omega \cup \{p\}$.
  Example: $f(z) = \frac{\sin z}{z}$ on $\Omega = \mathbb{C} \setminus \{0\}$.

$f(0) = 1$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\frac{\sin z}{z} = \underbrace{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots}$$

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Fact

- *("Picard's theorem") If f has an essential singularity in p, then on every $\mathcal{N}^*_{<\varepsilon}(p)$ it attains all possible values from $\mathbb{C}$, except at most one.*
- *If f has a pole in p, then $\lim_{z\to p} |f(z)| = +\infty$.*
- *If f has a removable singularity in p, then $\lim_{z\to p} f(z) \in \mathbb{C}$.*

Properties of poles

Proposition

A function f has a pole of degree d in p, iff it can be expressed, on some $\mathcal{N}^*_{<\varepsilon}(p)$, as

$$f(z) = \sum_{n=-d}^{\infty} a_n (z-p)^n$$
$$= \frac{a_{-d}}{(z-p)^d} + \frac{a_{-d+1}}{(z-p)^{d-1}} + \cdots + \frac{a_{-1}}{z-p} + a_0 + a_1(z-p) + a_2(z-p)^2 + \cdots$$

with $a_{-d} \neq 0$.

Note: A series of the form $\sum_{n=-\infty}^{\infty} a_n(z-p)^n$ is known as Laurent series.

Pf: "$\Rightarrow$" f has pole of deg d: $g(z) = f(z)(z-p)^d = \sum_{n=0}^{\infty} a_n^* (z-p)^n$, hence $f(z) = \sum_{n=0}^{\infty} a_n^* (z-p)^{n-d}$

where $a_n^* = a_{n-d}$

"$\Leftarrow$" $f(z) = \sum_{n=-d}^{\infty} a_n (z-p)^n \Rightarrow f(z)(z-p)^d$ is analytic in p. $\square$

Poles and zeros

Definition

A function g analytic in a point p has a zero of order d (a.k.a. degree d, or multiplicity d) in p, if it can be expressed, on some $\mathcal{N}_{<\varepsilon}(p)$, as

$(d \in \mathbb{N})$

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Proposition

A function g has a zero of degree $d > 0$ in p iff $\frac{1}{g}$ has a pole of degree d in p.

Pf: g has zero of deg. d $\iff$ $g(z) = h(z) \cdot (z-p)^d$, where $h(z) \neq 0$ and ~~an~~ h analytic in p

$\iff$ $\dfrac{1}{g(z)} = \dfrac{1}{(z-p)^d} \cdot \underbrace{\dfrac{1}{h(z)}}_{\text{analytic in } p \text{ an nonzero in } p}$ $\iff$ $\dfrac{1}{g}$ has a pole of deg. d in p.

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Proposition

Let f be meromorphic on a domain Ω, and suppose it has only finitely many poles in Ω. Then there is a rational function $R(z)$ such that the function $g(z) = f(z) - R(z)$ has an analytic continuation to Ω.