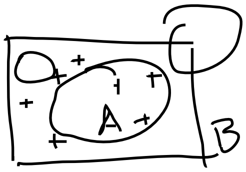
1. Show that the mixed binomial point process with the Poisson distribution (with parameter λ) of the number of points N is a Poisson process with the intensity measure $\lambda \frac{\nu(\cdot)}{\nu(B)}$.

Ψ ... binomial p.p. $\underset{\text{fixed}}{\underline{n}}$ points iid in $B \in \mathcal{B}(E)$ according to ν



(n, B, ν)

$$P(\Psi(A) = k) = \binom{n}{k}\left(\frac{\nu(A)}{\nu(B)}\right)^k\left(1 - \frac{\nu(A)}{\nu(B)}\right)^{n-k}, \quad k = 0, \dots, n$$

$A \subseteq B$

Φ ... mixed binomial p.p.: $N \sim Po(\lambda)$, conditionally on $N = n$ the process is binomial with n points

Poisson p.p.: 1) $\Phi(A) \sim Po(\Lambda(A)) \quad \forall A \in \mathcal{B}(E)$

Λ ... intensity measure

2) $\underset{\text{disjoint}}{A_1, \dots, A_k} \in \mathcal{B}(E)$: $\Phi(A_1), \dots, \Phi(A_k)$ inde

ad 1) $A \subseteq B$ ... otherwise take $A' = A \cap B$

take $k \in \mathbb{N}_0$, $\displaystyle P(\Phi(A) = k) = \sum_{n=k}^{\infty} P(\Phi(A) = k, N = n) =$

$$= \sum_{n=k}^{\infty} P(\Phi(A) = k \mid N = n) \cdot P(N = n) =$$

$$= \sum_{n=k}^{\infty} \binom{n}{k}\left(\frac{\nu(A)}{\nu(B)}\right)^k\left(1 - \frac{\nu(A)}{\nu(B)}\right)^{n-k} \cdot e^{-\lambda} \frac{\lambda^n}{n!} =$$

$$= \left(\frac{\nu(A)}{\nu(B)}\right)^k e^{-\lambda} \cdot \lambda^k \cdot \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} \cdot \frac{\lambda^{n-k}}{\cancel{n!}} \cdot \left(1 - \frac{\nu(A)}{\nu(B)}\right)^{n-k}$$

$$\dots \sum_{n=k}^{\infty} \frac{1}{(n-k)!} \lambda^{n-k} \left(1 - \frac{\nu(A)}{\nu(B)}\right)^{n-k}$$

$l = n - k$

$$= \exp\left\{\lambda\left(1 - \frac{\nu(A)}{\nu(B)}\right)\right\}$$

$$= \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} \cdot e^{-\lambda \frac{\nu(A)}{\nu(B)}} \left(\lambda \frac{\nu(A)}{\nu(B)}\right)^k \frac{1}{k!}$$

$\dots Po\left(\lambda \frac{\nu(A)}{\nu(B)}\right)$

ad 2) $A_1, A_2 \subseteq B$ disjoint , $\ell_1, \ell_2 \in \mathbb{N}_0$

$$\mathbb{P}(\Phi(A_1) = \ell_1, \Phi(A_2) = \ell_2) \overset{?}{=} \mathbb{P}(\Phi(A_1) = \ell_1) \cdot \mathbb{P}(\Phi(A_2) = \ell_2)$$

$$\| $$

$$\sum_{n=\ell_1+\ell_2}^{\infty} \mathbb{P}(\Phi(A_1) = \ell_1, \Phi(A_2) = \ell_2 \mid N = n)\, \mathbb{P}(N=n) =$$

$$= \sum_{n=\ell_1+\ell_2}^{\infty} \frac{n!}{\ell_1!\,\ell_2!\,(n-\ell_1-\ell_2)!} \cdot \left(\frac{\nu(A_1)}{\nu(B)}\right)^{\ell_1} \left(\frac{\nu(A_2)}{\nu(B)}\right)^{\ell_2} \left(1 - \frac{\nu(A_1)+\nu(A_2)}{\nu(B)}\right)^{n-\ell_1-\ell_2}$$

$\hookrightarrow$ multinomial distribution $\cdot\, e^{-\lambda} \frac{\lambda^n}{n!}$

$$= \frac{1}{\ell_1!} \frac{1}{\ell_2!} \left(\frac{\nu(A_1)}{\nu(B)}\right)^{\ell_1} \left(\frac{\nu(A_2)}{\nu(B)}\right)^{\ell_2} \cancel{e^{-\lambda}} \cdot \lambda^{\ell_1+\ell_2} \cdot$$

$$\cdot \underbrace{\sum_{n=\ell_1+\ell_2}^{\infty} \frac{1}{(n-\ell_1-\ell_2)!} \lambda^{n-\ell_1-\ell_2} \left(1 - \frac{\nu(A_1)+\nu(A_2)}{\nu(B)}\right)^{n-\ell_1-\ell_2}}_{\exp\left\{\lambda\left(\cancel{1}- \frac{\nu(A_1)+\nu(A_2)}{\nu(B)}\right)\right\}}$$

$$\rightsquigarrow \exp\left\{-\lambda \frac{\nu(A_1)}{\nu(B)}\right\} \cdot \exp\left\{-\lambda \frac{\nu(A_2)}{\nu(B)}\right\}$$

$$= \left(\frac{1}{\ell_1!}\left(\lambda \frac{\nu(A_1)}{\nu(B)}\right)^{\ell_1} \exp\left\{-\lambda \frac{\nu(A_1)}{\nu(B)}\right\}\right) \left(\frac{1}{\ell_2!}\left(\lambda \frac{\nu(A_2)}{\nu(B)}\right)^{\ell_2} \exp\left\{-\lambda \frac{\nu(A_2)}{\nu(B)}\right\}\right)$$

... the same works for n-tuples of disjoint sets:

$$\mathbb{P}(\Phi(A_1) = \ell_1, \dots, \Phi(A_n) = \ell_n \mid N = m) =$$

$$= \frac{m!}{\ell_1!\,\ell_2! \dots \ell_n!\,(m-\ell_1-\ell_2-\dots-\ell_n)!} \left(\frac{\nu(A_1)}{\nu(B)}\right)^{\ell_1} \dots \left(\frac{\nu(A_n)}{\nu(B)}\right)^{\ell_n} \left(1 - \frac{\nu(A_1)+\dots+\nu(A_n)}{\nu(B)}\right)$$