Analytic combinatorics

Lecture 5

April 7, 2021

Global properties of analytic functions

Recall: A complex function f is analytic in $z_0 \in \mathbb{C}$, if it is equal to the sum of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ on a neighborhood of z_0.

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Let $\Omega \subseteq \mathbb{C}$ be an open set. We say that f is analytic on Ω, if f is analytic in every point of Ω.

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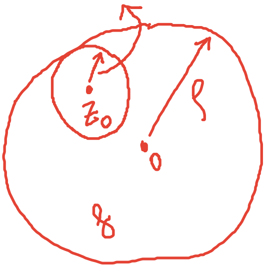
Proposition

Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $\rho > 0$. Define a function $f : \mathcal{N}_{<\rho}(0) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then f is analytic on $\mathcal{N}_{<\rho}(0)$. Moreover, for $z_0 \in \mathcal{N}_{<\rho}(0)$, the series expansion of f with center z_0 has radius of convergence at least $\rho - |z_0|$.



Idea: $f(z) = \sum_{n=0}^{\infty} a_n z^n =$

$$= \sum_{n=0}^{\infty} a_n (z - z_0 + z_0)^n =$$

$$= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k \overset{?}{=}$$

$$= \sum_{k=0}^{\infty} \Big(\underbrace{\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}}_{b_k} \Big) (z - z_0)^k$$

Definition

Let $f\colon M \to \mathbb{C}$ be a function, let Ω be an open set with $M \subseteq \Omega$. A function $g\colon \Omega \to \mathbb{C}$ is an analytic continuation of f if

- for every $z \in M$, $f(z) = g(z)$, and
- g is analytic on Ω.

Example:

$A(x) = 1 + x + x^2 + x^3 + \dots$, $\rho = 1$, $f(z) = 1 + z + z^2 + \dots = \boxed{\frac{1}{1-z}}$ on $\underbrace{\mathcal{U}(0)}_{<1} = M$

$g(z) = \frac{1}{1-z}$

$z \in \mathbb{C}\setminus\{1\}$

g is analytic on $\mathbb{C}\setminus\{1\}$

because $1 - z$ is analytic on $\mathbb{C}$

$f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ may converge for $z \notin \mathcal{U}_{\le 1}(0)$

Remark: $g = \frac{1}{1-z}$ has no continuation on $\mathbb{C}$ because it is ~~a~~ unbounded in every $\mathcal{U}_{<\varepsilon}(1)$

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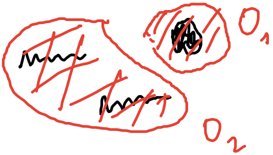
A set $X \subseteq \mathbb{C}$ is ...

- open if for every $z \in X$ there is an $\varepsilon > 0$ such that $\mathcal{N}_{<\varepsilon}(z) \subseteq X$;
- disconnected if there are two disjoint open sets O_1 and O_2 such that $O_1 \cap X \neq \emptyset$, $O_2 \cap X \neq \emptyset$, and $X \subseteq O_1 \cup O_2$; otherwise, the set X is connected;



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- a domain if it is nonempty, open and connected;
- discrete if for every $z \in X$ there is $\varepsilon > 0$ such that $\mathcal{N}_{<\varepsilon}(z) \cap X = \{z\}$.

ex: $X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is discrete

$X \cup \{0\}$ is not discrete

Global analytic uniqueness

Recall: If f is analytic in $z \in \mathbb{C}$ with $f(z) = 0$, then either f is identically zero on a neighborhood of z, or f is never zero on a punctured neighborhood of z.

Proposition (Global analytic uniqueness)

Let f be a function analytic on a domain Ω. Then the set $Z_f = \{z \in \Omega;\ f(z) = 0\}$ is either discrete or equal to Ω.

Proof:

$$\mathcal{O}_1 := \{ z \in \Omega : f \text{ is identically } 0 \text{ on a } \mathcal{N}_{<\varepsilon}(z) \text{ for } \varepsilon > 0 \}$$

$$\mathcal{O}_2 := \{ z \in \Omega : f \text{ is never } 0 \text{ on a } \mathcal{N}^*_{<\varepsilon}(z), \text{ for some } \varepsilon > 0 \}$$

$\Omega = \mathcal{O}_1 \cup \mathcal{O}_2$, clearly $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$

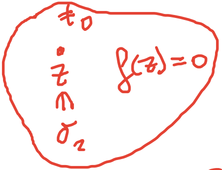
$\mathcal{O}_1$ is open, $\mathcal{O}_2$ too

Global analytic uniqueness (proof)

But Ω is connected, ~~ke~~ hence

$$\mathcal{O}_1 = \emptyset \quad \text{or} \quad \mathcal{O}_2 = \emptyset.$$

If $\mathcal{O}_1 = \emptyset$, then $\mathcal{O}_2 = \Omega$ and $Z_f = \{z \in \Omega,\ f(z) = 0\}$ is discrete (possibly empty)



If $\mathcal{O}_2 = \emptyset$, then $\mathcal{O}_1 = \Omega$ and $Z_f = \Omega$. $\square$

Global analytic uniqueness – consequences

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Let f and g be two functions analytic on a domain Ω. Then the set $\{z \in \Omega;\ f(z) = g(z)\}$ is either discrete or equal to Ω.

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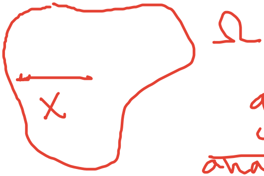
Corollary

Let $X \subseteq \mathbb{C}$ be a set which is not discrete, let $f : X \to \mathbb{C}$ be a function, let Ω be a domain containing X as a subset. Then f has at most one analytic continuation of to Ω.



If $g_1 : \Omega \to \mathbb{C}$ and $g_2 : \Omega \to \mathbb{C}$ are both ~~analytic~~ analytic cont. of f, they agree on X, hence they agree on Ω

Some notable analytic functions

Definition

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

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- For any $z \in \mathbb{C}$, we have

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2},$$
$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i},$$
$$\exp(iz) = \cos(z) + i \sin(z).$$

Properties of sin, cos and exp

Proposition

The following holds:

$\exp(iz) = \cos(z) + i\sin(z)$

- a) For $k \in \mathbb{Z}$ and $z \in \mathbb{C}$: $\exp(kz) = (\exp(z))^k$.
- For $z \in \mathbb{C}$: $\sin(z + 2\pi) = \sin(z)$, $\cos(z + 2\pi) = \cos(z)$, $\exp(z + 2\pi i) = \exp(z)$.
- For $z \in \mathbb{C}$: $\sin^2(z) + \cos^2(z) = 1$.
- b) For $w, z \in \mathbb{C}$: $\exp(w + z) = \exp(w)\exp(z)$.

Proof: We know that the equalities hold for $z, w \in \mathbb{R}$.

a) $\exp(kz)$ is analytic on $\mathbb{C}$, so is $(\exp(z))^k$, they agree on $\mathbb{R}$, which is not discrete

$\Rightarrow \exp(kz) = (\exp(z))^k$ on $\mathbb{C}$

b) for $w \in \mathbb{R}$: $\forall z \in \mathbb{R}$: $\exp(w+z) = \exp(w)\exp(z)$

$\Rightarrow \forall z \in \mathbb{C}$:

for fixed z: $\forall w \in \mathbb{R}$ $=$ holds $\Rightarrow \forall w \in \mathbb{C}$: $=$ holds $\square$

Combining analytic continuations

Proposition

Let $X \subseteq \mathbb{C}$ be a set which is not discrete, let $f : X \to \mathbb{C}$ be a function, let Ω_1 and Ω_2 be two domains, with $X \subseteq \Omega_1 \cap \Omega_2$. Let g_1 and g_2 be the analytic continuation of f to Ω_1 and Ω_2, respectively. If $\Omega_1 \cap \Omega_2$ is connected, then g_1 and g_2 agree on $\Omega_1 \cap \Omega_2$, and together form an analytic continuation of f to $\Omega_1 \cup \Omega_2$.

Note: The assumption that $\Omega_1 \cap \Omega_2$ is connected is essential.

$\Omega_1 \cap \Omega_2$ connected

$\Rightarrow$ it is a domain

$\Rightarrow$ f has at most one continuation to $\Omega_1 \cap \Omega_2$ $\Rightarrow$ g_1 and g_2 agree on $\Omega_1 \cap \Omega_2$

$\square$

Complex square root

Goal: Let us look for an inverse function to $f(z) = z^2$.

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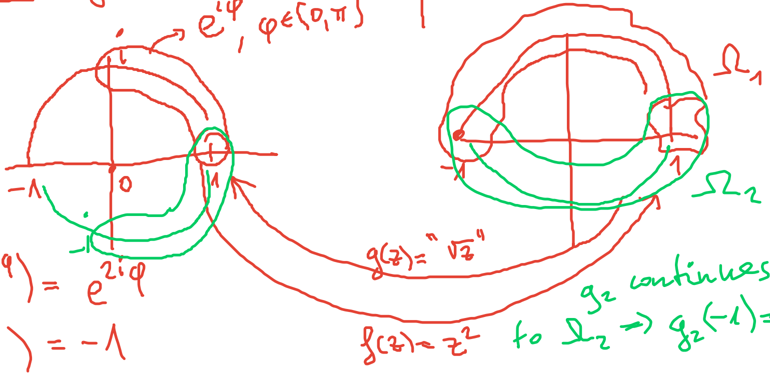
Recall: If f is analytic in $z_0 \in \mathbb{C}$ and $f'(z_0) \neq 0$, then f is maps a neighborhood $\mathcal{N}_{<\varepsilon}(z_0)$ of z_0 bijectively to an open set Ω, and its inverse function $f^{\langle -1 \rangle} : \Omega \to \mathcal{N}_{<\varepsilon}(z_0)$ is analytic in $f(z_0)$.

$f'(z) = 2z \neq 0$ on $\mathbb{C} \setminus \{0\}$

$z = 1$: $f(1) = 1$

g_1 continues g to Ω_1 $\Rightarrow$ $g_1(-1) = i$

$e^{i\varphi}$, $\varphi \in [0, \pi]$



$f(e^{i\varphi}) = e^{2i\varphi}$

$f(i) = -1$

g_2 continues g to Ω_2 $\Rightarrow$ $g_2(-1) = -i$

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Conclusion: $f(z) = z^2$ has an analytic inverse in a neighborhood of any $z_0 \neq 0$, but this cannot be analytically continued to $\mathbb{C} \setminus \{0\}$.

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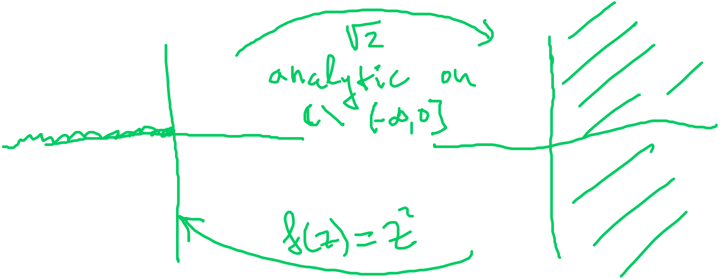
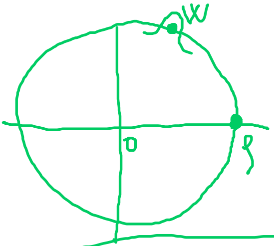
Definition

For any $z \in \mathbb{C} \setminus (-\infty, 0]$ we let $\sqrt{z}$ denote the unique number w with $\Re(w) > 0$ satisfying $w^2 = z$.



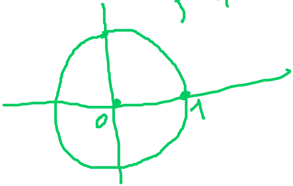
Fact (Pringsheim, Vivanti; 1890's)

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $\rho \in (0, +\infty)$, and let us define $f : \mathcal{N}_{<\rho}(0) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then there is at least one point w with $|w| = \rho$ such that f has no analytic continuation to any domain containing w. If we additionally assume that $a_n \geq 0$ for all n, then the conclusion holds for $w = \rho$.



$f(z) = 1 + z + z^2 + z^3 + \ldots$

$\rho = 1$



so $\sum a_n x^n$ has exp. growth $= \frac{1}{\rho} = 1$

$g(z) = \frac{1}{1-z}$ on $\mathbb{C} \setminus \{1\}$

$g(z) = \sum_{n=0}^{\infty} a_n z^n$ has $\rho = 1$

Example