6. Let Φ be a simple point process. Check that the following formulas hold for $B, B_1, B_2, B_3 \in \mathcal{B}$:

a) $M^{(2)}(B_1 \times B_2) = \Lambda(B_1 \cap B_2) + \alpha^{(2)}(B_1 \times B_2)$,

b) $M^{(3)}(B_1 \times B_2 \times B_3) = \Lambda(B_1 \cap B_2 \cap B_3) + \alpha^{(2)}((B_1 \cap B_2) \times B_3) + \alpha^{(2)}((B_1 \cap B_3) \times B_2) + \alpha^{(2)}((B_2 \cap B_3) \times B_1) + \alpha^{(3)}(B_1 \times B_2 \times B_3)$,

c) $\alpha^{(n)}(B \times \cdots \times B) = \mathbb{E}[\Phi(B)(\Phi(B) - 1) \cdots (\Phi(B) - n + 1)]$. ≥ 0 $\quad \forall n \in \mathbb{N} \ \forall B \in \mathcal{B}(E)$

Campbell theorem: $M^{(2)}(B_1 \times B_2) = \mathbb{E} \sum_{x_1, x_2 \in \operatorname{supp} \Phi} \mathbb{1}[x_1 \in B_1, x_2 \in B_2]$

$$\alpha^{(2)}(B_1 \times B_2) = \mathbb{E} \sum^{\neq}_{x_1, x_2 \in \operatorname{supp} \Phi} \mathbb{1}[x_1 \in B_1, x_2 \in B_2]$$

"$\neq$ means $x_1 \neq x_2$"

a) $$M^{(2)}(B_1 \times B_2) = \alpha^{(2)}(B_1 \times B_2) + \mathbb{E} \sum_{x_1 \in \operatorname{supp} \Phi} \mathbb{1}[x_1 \in B_1, x_1 \in B_2] =$$
$$= \alpha^{(2)}(B_1 \times B_2) + \mathbb{E} \sum_{x \in \operatorname{supp} \Phi} \mathbb{1}[x \in B_1 \cap B_2] = \alpha^{(2)}(B_1 \times B_2) + \Lambda(B_1$$

b) $$M^{(3)}(B_1 \times B_2 \times B_3) = \alpha^{(3)}(B_1 \times B_2 \times B_3) + \mathbb{E} \sum_{x_1 \in \operatorname{supp} \Phi} \mathbb{1}(x_1 \in B_1, x_1 \in B_2, x_1 \in B_3)$$
$$+ \mathbb{E} \sum^{\neq}_{x_1, x_2} \mathbb{1}(x_1 \in B_1, x_2 \in B_2, x_1 \in B_3) + \mathbb{E} \sum^{\neq}_{x_1, x_2} \mathbb{1}(x_1 \in B_1, x_2 \in B_2, x_2$$

"$x_3 = x_1$"

$$+ \mathbb{E} \sum^{\neq}_{x_1, x_2} \mathbb{1}(x_1 \in B_1, x_1 \in B_2, x_2 \in B_3) =$$
$$= \alpha^{(3)}(B_1 \times B_2 \times B_3) + \Lambda(B_1 \cap B_2 \cap B_3) + \alpha^{(2)}((B_1 \cap B_3) \times B_2)$$
$$+ \alpha^{(2)}(B_1 \times (B_2 \cap B_3)) + \alpha^{(2)}((B_1 \cap B_2) \times B_3)$$
$$\alpha^{(2)}((B_2 \cap B_3) \times B_1) \quad \text{--- symmetry}$$

$$\alpha^{(m)}(B\times\dots\times B) \overset{\text{Campbell}}{=} \mathbb{E}\sum^{\neq}_{X_1,\dots,X_n\in \operatorname{supp}\Phi} \mathbb{1}(X_1\in B,\dots,X_n\in B) = \mathbb{E}\ \text{„\# of n-tuples in}$$

$$= (*)$$

$$\overset{?}{=}$$

$$\mathbb{E}\left[\Phi(B)(\Phi(B)-1)\cdots(\Phi(B)-n+1)\right]$$

$$\alpha^{(2)}(B\times B) = \underbrace{M^{(2)}(B\times B)}_{\mathbb{E}\Psi(B)^2} - \underbrace{\Lambda(B\cap B)}_{\Lambda(B)=\mathbb{E}\Psi(B)} = \mathbb{E}\left[\Psi(B)^2-\Psi(B)\right] =$$

$n=2$

$$= \mathbb{E}\left[\Psi(B)(\Psi(B)-1)\right]$$

$$(*) = \mathbb{E}\sum^{\neq}_{X_1,\dots,X_{n-1}} \mathbb{1}[X_1\in B,\dots,X_{n-1}\in B]\cdot\underbrace{\Bigg(\sum_{\substack{X_n\in\operatorname{supp}\Phi\\ X_n\notin\{X_1,\dots,X_{n-1}\}}}\mathbb{1}(X_n\in B)\Bigg)}$$

$\hookrightarrow$ we already know $X_1\in B,\dots,X_{n-1}\in B$, $\quad \Phi(B)-(n-1) =$

= „how many points are in B, after $X_1,\dots,X_{n-1}$ removed?"

$\hookrightarrow$ Proceed the same with X_{n-1} etc.

$$(**) = \mathbb{E}\big(\text{number of n-tuples of distinct points from } \Phi(B)$$

$$= \mathbb{E}\binom{\Phi(B)}{n}\cdot n! = \mathbb{E}\,\Phi(B)(\Phi(B)-1)\cdots(\Phi(B)-n+1).$$