NMAI059 Probability and statistics 1
Class 6

Robert Šámal

Overview

Random vectors

Conditional distribution

Continuous random variables

Particular continuous distributions and their parameters

What have we learned

- Joint PMF: $p_{X,Y}(x,y) = P(X = x \,\&\, Y = Y)$
- Example: multinomial distribution
- Marginal PMF: $p_X(x) = \sum_{y \in Im(y)} p_{X,Y}(x,y)$
- Example: coupling
- X, Y are independent iff
  $P(X = x \,\&\, Y = y) = P(X = x)P(Y = y)$
  That is, iff $p_{X,Y}(x,y) = p_X(x)p_Y(y)$.
- If X, Y are independent then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.
- $\mathbb{E}(g(X,Y)) = \sum_{x \in ImX} \sum_{y \in ImY} g(x,y)P(X = x, Y = y)$
- **Linearity of expectation** For any r.v.s X, Y and $a, b \in \mathbb{R}$ we have $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$.
- **convolution formula**
  $P(X + Y = n) = \sum_{k \in Im(X)} P(X = k, Y = n - k)$

uniquely

$P_{X,Y} \rightarrow P_X$

P_Y

not unique

Overview

Conditional PMF

X, Y – discrete random variables on $(\Omega, \mathcal{F}, P)$, $A \in \mathcal{F}$

- $p_{X|A}(x) := P(X = x \mid A)$
  example: X is outcome of a roll of a die, $A =$ we got an even number
- $p_{X|Y}(x|y) = P(X = x \mid Y = y)$ example: X, Z is an outcome of two independent die rolls, $Y = X + Z$.

$$p_{X|Y}(6|10) =$$

- $p_{X|Y}$ from $p_{X,Y}$:

$$p_{X|Y}(x,y) = \frac{P(X=x \,\&\, Y=y)}{P(Y=y)} = \frac{p_{X,Y}(x,y)}{\sum_{x'} p_{X,Y}(x',y)}$$

Joint vs. conditional PMF

$X, Z \sim \text{Uniform}\{1, 2, \dots, 6\}$, $Y = X + Z$

Y $11 = 5+6 = 6+5$

scale each column so that it has sum 1

$p_{X,Y}$	...	10	11	12
1		0	0	0
2		0	0	0
3		0	0	0
4		$\frac{1}{36}$	0	0
5		$\frac{1}{36}$	$\frac{1}{36}$	0
6		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
Σ		$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

X

$p_{X\|Y}$	...	10	11	12
1		0	0	0
2		0	0	0
3		0	0	0
4		$\frac{1}{3}$	0	0
5		$\frac{1}{3}$	$\frac{1}{2}$	0
6		$\frac{1}{3}$	$\frac{1}{2}$	1
Σ		1	1	1

x, y

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$p_{X,Y}(x,y) = \underbrace{p_Y(y)}_{p_Y} p_{X|Y}(x|y)$$

$p_{X|Y}(x|y)$ given $Y = 11$: $X = 5$ or 6, both w. prob. $\frac{1}{2}$

Ex. Splitting the Poisson

$X \sim \text{Pois}(\lambda)$ ..... # emails in a day

Y is the # of spams among these X emails

each of the emails has prob. p to be a spam (indep. of others)

$Z = X - Y$ -- # of non-spams (hams) among the X emails

$$p_{Y|X}(k|n) = P(Y=k \mid X=n) = (\text{binom. distr.}) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \le k \le n$$

$$p_X(n) = P(X=n) = (\text{Poisson}) = \frac{\lambda^n}{n!} e^{-\lambda}$$

$n \in \mathbb{N}_0, \quad 0 \le k \le n$

$$p_{X,Y}(n,k) = p_X(n) \cdot p_{Y|X}(k|n) = \frac{\lambda^n}{n!} e^{-\lambda} \binom{n}{k} p^k (1-p)^{n-k} = \lambda^n e^{-\lambda} \frac{p^k}{k!} \frac{(1-p)^{n-k}}{(n-k)!}$$

$\frac{n!}{k!(n-k)!}$

$$p_Y(k) = \sum_{n \ge k} p_{X,Y}(n,k) = \sum \lambda^n e^{-\lambda} \frac{p^k}{k!} \frac{(1-p)^{n-k}}{(n-k)!} = \sum_{\ell \ge 0} \lambda^{k+\ell} e^{-\lambda} \frac{p^k}{k!} \frac{(1-p)^\ell}{\ell!} = (*)$$

$\ell = n-k \ge 0$

$$(*) = \lambda^k e^{-\lambda} \frac{p^k}{k!} \sum_{\ell \ge 0} \frac{[\lambda(1-p)]^\ell}{\ell!} = e^{\lambda(1-p)} \cdot \ldots = \frac{(p\lambda)^k}{k!} e^{-p\lambda}$$

$$\boxed{Y \sim \text{Pois}(p\lambda)}$$

$$\boxed{Z \sim \text{Pois}((1-p)\lambda) \ \& \ Y, Z \text{ are indep.}}$$

$$P(Y=k \ \& \ Z=n-k) = P(Y=k \ \& \ X=n) = \lambda^n e^{-\lambda} \frac{p^k}{k!} \frac{(1-p)^{n-k}}{(n-k)!}$$

$$P(Y=k) = \frac{(p\lambda)^k}{k!} e^{-p\lambda}$$

$\sim Pois(\lambda p)$

$$P(Z=n-k) = \frac{((1-p)\lambda)^{n-k}}{(n-k)!} e^{-(1-p)\lambda}$$

$\sim Pois((1-p)\lambda)$

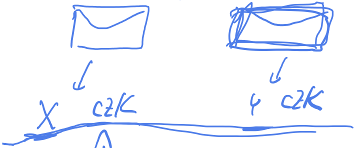
$\times$

$$\lambda^k \cdot \lambda^{n-k} = \lambda^n$$

$\longrightarrow$ Y, Z are independent

Two envelope paradox

$(X, Y > 0)$



> Either
> $X = 2Y$ or $Y = 2X$ with equal prob.

Take X, find $X = x$. Then either $Y = \frac{x}{2}$ or $Y = 2x$

$$E(Y \mid X = x) = \frac{1}{2}\cdot\frac{x}{2} + \frac{1}{2}\cdot 2x = \frac{5}{4}x > x \; !$$

So, whatever value x, we should switch! We did not need to know x! THIS MUST BE WRONG!

Problem 1 We did not specify joint ~~PDF~~. With prob. 2^{-k} we put 2^{k} CZK in one env. & 2^{k+1} CZK in the other (choosing envel. at random).

$$P_{X,Y}(2^{k}, 2^{k+1}) = P_{X,Y}(2^{k+1}, 2^{k}) = \frac{1}{2} 2^{-k} \quad (k = 1, 2, 3, \ldots)$$

If $X = 1$ then necessary $Y = 2$, so we should switch.

$$p_{X,Y}(2^k, 2^{k+1}) = p_{X,Y}(2^{k+1}, 2^k) = \frac{1}{2} 2^{-k}$$ we spec'd joint PMF

If $X = 2^n$, what should we do? $n \geq 1$

$Y = 2^{n+1}$ w. prob. $p_{X,Y}(2^n, 2^{n+1}) = \frac{1}{2} 2^{-(n+1)} = 2^{-n-2}$

or

$Y = 2^{n-1}$ w. prob. $p_{X,Y}(2^n, 2^{n-1}) = \frac{1}{2} 2^{-n} = 2^{-(n+1)}$

$$P(X = 2^n) = \sum_y P(X=2^n, Y=y) = \frac{3}{\cancel{2}\,4} 2^{-n}$$

marginal PMF

conditional PMF

$$E(Y \mid X = 2^n) = \sum_y y \cdot P(Y = y \mid X = 2^n)$$

$$= 2^{n+1} \cdot \frac{\cancel{2^{-n}}\,\frac{1}{4}}{\frac{3}{\cancel{2}\,4} \cancel{2^{-n}}} + 2^{n-1} \frac{\frac{1}{2} \cancel{2^{-n}}}{\frac{3}{\cancel{2}\,4} \cancel{2^{-n}}}$$

$$EX = \sum_{n=1} 2^n \cdot \frac{3}{\cancel{2}\,4} 2^{-n} = \infty$$

$$\| \quad EY$$

$$= 2^{n+1} \cdot \frac{\cancel{2}\,1}{3} + 2^{n-1} \cdot \frac{2}{3} = 2^{n-1}\left(\frac{\cancel{8}\,4}{3} + \frac{2}{3}\right) = \cancel{3}\,2 \cdot 2^{n-1} = 2^n \quad \cancel{= \frac{3}{2} 2^n > 2^n}$$

~~$E(X \mid X = 2^n) = 2^n$ ⟶ This seems to imply $EY > EX$, so we should switch?? ...~~

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General random variable

Definition

Random variable on $(\Omega, \mathcal{F}, P)$ is a mapping $X : \Omega \to \mathbb{R}$, such that for each $x \in \mathbb{R}$

$$\{\omega \in \Omega : X(\omega) \le x\} \in \mathcal{F}.$$

we want to measure $P(X \le x)$

- discrete r.v. is a r.v.

$$\{\omega : X(\omega) \le x\} = \bigcup_{\substack{x' \in \mathrm{Im}(X) \\ x' \le x}} \underbrace{\{\omega : X(\omega) = x'\}}_{\in \mathcal{F} \text{ by def.}}$$

countable union — as X is discrete

$$P(X \le x) = \sum_{\substack{x' \le x \\ x' \in \mathrm{Im}(X)}} P(X = x')$$

CDF

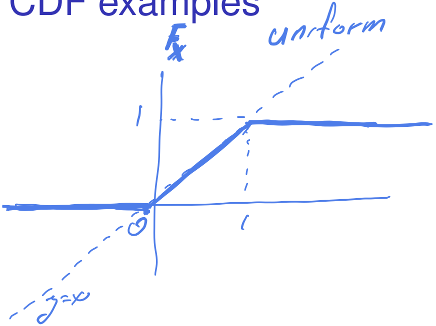
Definition

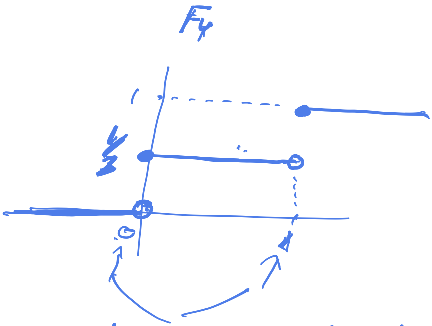
Cumulative distribution function, CDF of a r.v. X is a function

$$F_X(x) := P(X \leq x) = P(\{\omega \in \Omega : X(\omega) \leq x).$$

- F_X is a nondecreasing function
- $\lim_{x \to -\infty} F_X(x) = 0$
- $\lim_{x \to +\infty} F_X(x) = 1$
- F_X is right-continuous

CDF examples





here is all the action

$\{\mathrm{Im}(Y)\} = \{0, 1\}$

$F_X(x) = P(X \le x)$

$F_X(\frac{1}{3}) = P(X \le \frac{1}{3}) = \frac{1}{3}$

$Y = \begin{cases} 0 & \text{w.prob. } \frac{1}{2} \\ 1 & \frac{1}{2} \end{cases}$

Quantile function

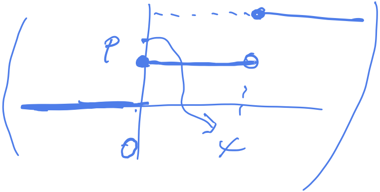
For a r.v. X we define its *quantile function* $Q_X : [0,1] \to \mathbb{R}$ by

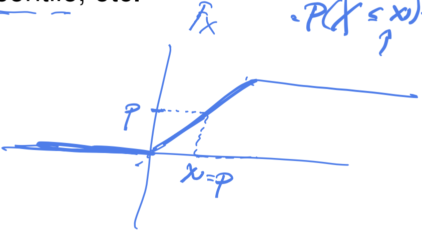
$$Q_X(p) := \min\{x \in \mathbb{R} : p \leq F_X(x)\}$$

- If F_X is continuous and increasing then $Q_X = F_X^{-1}$.
- $Q_X(1/2) =$ median (watch out if F_X is not strictly increasing!)
- $Q_X(10/100) =$ tenth percentile, etc.

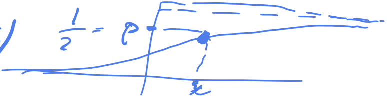
$F_X(x_0) = \frac{1}{2}$

$P(X \leq x_0) = \frac{1}{2}$





~~quart~~ quantiles $Q(\frac{1}{5}), Q(\frac{2}{5}), Q(\frac{3}{5})$



Continuous random variable

Definition

R.v. X is called continuous, if there is nonnegative real function f_X such that $\forall x \in \mathbb{R}$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)dt.$$

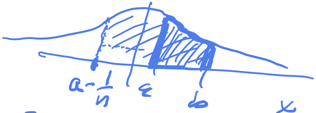
$\longrightarrow \int_{-\infty}^{\infty} f_X = 1$

(Sometimes such X is said to be absolutely continuous.)
Function f_X is called the probability density function, PDF of X.

- Alternatively: we pick a point from the probability space corresponding to the area under graph of f – nonnegative function with $\int_{-\infty}^{\infty} f = 1$. $P((X,Y) \in S) = \text{area of } S$
- Let (X, Y) denote the coordinates of the point. is calc.
- Then X is a random variable with PDF f.

To verify: $P(X \le x) = P((X,Y) \in S_x) = \text{area of } S_x = \int_{-\infty}^{x} f(t)dt$

Using density



$f \le 1$

Theorem

Let X be a continuous r.v. with PDF f_X. Then

1. $P(X = x) = 0$ *for every* $x \in \mathbb{R}$.
2. $P(a \le X \le b) = \int_a^b f_X(t)dt$ *for every* $a, b \in \mathbb{R}$.

$$\int_x^x f = 0$$

Proof (2)

$$P((X,Y) \in S_b \setminus S_a) = \text{area of } S_b \setminus S_a = \int_a^b f$$

$$P(a \le X \le b) = \lim_{n\to\infty} P\left(a - \tfrac{1}{n} < X \le b\right)$$

$$= \lim_{n\to\infty} \int_{a-\frac{1}{n}}^{b} f = \int_a^b f \Longrightarrow (1)$$

Expectation of a continuous r.v.

Definition

Consider a continuous r.v. X with PDf f_X. Then its expectation (expected value, mean) is denoted by $\mathbb{E}(X)$ and defined by

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \; f_X(x) dx,$$

whenever the integral is defined; that is unless it is a type $\infty - \infty$. TODO EXPLAIN?

- An analogy with computing a center of mass of a pole from a formula for its density.

Continuous LOTUS

Theorem (LOTUS)

Consider a continuous r.v. X with density f_X and a real function g. Then we have

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx,$$

whenever the integral is defined.

(We skip the proof.)

Variance of a continuous r.v.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \; f_X(x) dx$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \; f_X(x) dx$$

Writing $\mu = \mathbb{E}(X)$, we have

$$var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 \; f_X(x) dx = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$