M $\quad$ X omezený $\mathcal{F}_t$-pred.

↳ zpr. spojitý $\mathcal{F}_t$-mart. $|M_v| < \infty$ $\quad E|M_v| < \infty$ $\qquad \int X\,dM$ je $\mathcal{F}_t$-martingal

$E\int_0^T |X_s|\,d|M_v|(s) < \infty$ ale tato podmínka se těžko ověřuje.

$Y_t = \int_0^t X\,dM \qquad Y_t^2 \qquad EY_t^2 < \infty \quad \exists\ \langle Y,Y\rangle$ prediktabilní variace

$Y^2 - \langle Y,Y\rangle$ je martingal

$\downarrow$

Y má konstantní střední hodnotu, $\quad M_0 = 0 \Rightarrow Y_0 = 0$

$EY_t = 0 \qquad E(Y_t^2 - \langle Y,Y\rangle) \qquad E\langle Y,Y\rangle_t = EY_t^2 = \operatorname{var} Y_t$

Věta 9: Buďte H, G omezené $\mathcal{F}_t$-prediktabilní

M, N zprava spojité $\mathcal{F}_t$-martingaly $|M_t| < \infty$, $|N_t| < \infty$, $M_0 = N_0 = 0$

$E|M_t| < \infty$, $E|N_t| < \infty$, $E|M_t|^2 < \infty$, $E|N_t|^2 < \infty$ (L_2-martingaly)

Pak $\langle \int H\,dM, \int G\,dN \rangle_t \overset{s.j.}{=} \int_0^t HG\,d\langle M, N\rangle$

a speciálně $\langle \int H\,dM, \int H\,dM \rangle_t = \int_0^t H^2\,d\langle M, M\rangle$

Důkaz: ukážeme $E\left(\int_0^T H\,dM\right)^2 < \infty$

$|H| \leq k$ $\quad E\left(\int_0^T H\,dM\right)^2 \leq E\left(\int_0^T |H|\,d|M_t|\right)^2 \leq k^2 E|M_t|^2 < \infty$

N_t Poissonův s intenzitou $\lambda(t) > 0$

$N_t - \int_0^t \lambda(s)ds = N_t - \Lambda(t)$ je martingal $\qquad |(N-\Lambda)_t| = N_t + \Lambda(t) \qquad EN_t^2 < \infty$

$\Lambda(t) < \infty$

$\int H dM$ i $\int G dN$ jsou L_2-martingaly. Má smysl podívat se na jejich kovariaci:

prediktabilita?

Chceme dokázat, že $\underbrace{\int H dM}_{} \cdot \underbrace{\int G dN}_{} - \underline{\int HG \, d\langle M,N\rangle}$ je $\mathcal{F}_t$-martingal

souč. $\mathcal{F}_t$-martingalů

↑ proces s konečnou úplnou variací začínající v 0, $\mathcal{F}_t$-prediktabilní

$H, G, \langle M,N\rangle$ jsou $\mathcal{F}_t$-prediktabilní procesy

$\int H\,dM \int G\,dN - \int HG\,d\langle M,N\rangle$

adaptovanost – zřejmé

integrovatelnost $E\left|\int H\,dM \int G\,dN\right| \leq \Big(\underbrace{E\left(\int H\,dM\right)^2}_{<\infty} \underbrace{E\left(\int G\,dN\right)^2}_{<\infty}\Big)^{1/2}$

$$\left(E\langle M,N\rangle\right)^2 \leq E\langle M,M\rangle\, E\langle N,N\rangle$$

$$?\; E\left[\int_0^t H\,dM \int_0^t G\,dN - \int_0^s H\,dM \int_0^s G\,dN \,\middle|\, \mathcal{F}_s\right] \underset{?\,\text{s.j.}\,?}{=} E\left[\int_0^t HG\,d\langle M,N\rangle - \int_0^s HG\,d\langle M,N\rangle \,\middle|\, \mathcal{F}_s\right]$$

Pro elementární $M_0, N_0 = 0$, bez újmy na obecnosti předpokládejme stejné dělení pro H i G

$$\int_0^1 H\,dM = \sum_{i=0}^{n-1} \xi_i \left(M_{t_{i+1}\wedge 1} - M_{t_i\wedge 1}\right)$$

$$\int_0^1 G\,dM = \sum_{i=0}^{n-1} \eta_i \left(N_{t_{i+1}\wedge 1} - N_{t_i\wedge 1}\right)$$

$$\int_0^1 H\,dM = \int_0^s H\,dM + \int_s^1 H\,dM$$

$$\int_0^1 H\,dM \int_0^1 G\,dN - \int_0^s H\,dM \int_0^s G\,dN = \int_s^1 H\,dM \int_s^1 G\,dN + \int_0^s H\,dM \int_s^1 G\,dN + \int_s^1 H\,dM \int_0^s G\,dN$$

$$E\left[\int_s^t H\,dM \int_s^t G\,dN + \underbrace{\int_0^s H\,dM}\int_s^t G\,dN + \int_s^t H\,dM \underbrace{\int_0^s G\,dN} \,\Big|\, \mathcal{F}_s\right] =$$

$$= E\left[\int_s^t H\,dM \int_s^t G\,dN \,\Big|\, \mathcal{F}_s\right] + \int_0^s H\,dM \; E\left[\int_s^t G\,dN \,\Big|\, \mathcal{F}_s\right] + 0$$

přírůstek martingalu
v úseku $(s,t]$
a podmiňujeme $\mathcal{F}_s$
$= 0$

$$= E\left[\int_s^t H\,dM \int_s^t G\,dN \,\Big|\, \mathcal{F}_s\right]$$

Předpokládejme $s = t_l \quad t = t_k \qquad (l < k)$

$$\int_s^t H\,dM = \sum_{i=l}^{k-1} \xi_i \left(M_{t_{i+1}} - M_{t_i}\right)$$

$$\int_s^t G\,dN = \sum_{i=l}^{k-1} \eta_i \left(N_{t_{i+1}} - N_{t_i}\right)$$

$$E\left[\int_s^t H\,dM \int_s^t G\,dN \,\Big|\, \mathcal{F}_s\right] = E\left[\sum_{i=l}^{k-1} \xi_i \eta_i \left(M_{t_{i+1}} - M_{t_i}\right)\left(N_{t_{i+1}} - N_{t_i}\right) \Big|\, \mathcal{F}_{t_l}\right] =$$

$$+\, \cancel{E\left[\sum_{\substack{i \neq j \\ i,j=l}}^{k-1} \xi_i \eta_j \left(M_{t_{i+1}} - M_{t_i}\right)\left(N_{t_{j+1}} - N_{t_j}\right) \Big|\, \mathcal{F}_{t_l}\right]} = 0$$

$$E\Big(\qquad\qquad \big|\, \mathcal{F}_{t_i \vee t_j}\Big)$$

$$= E\left[\sum_{i=\ell}^{k-1} \xi_i \eta_i \underline{E\left((M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) \middle| \mathcal{F}_{t_i} \right)} \middle| \mathcal{F}_{t_\ell} \right] = \circledast$$

$M \cdot N - \langle M, N \rangle$ je martingal

$$E\left[(M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) \middle| \mathcal{F}_{t_i} \right] = E\left[M_{t_{i+1}} N_{t_{i+1}} - M_{t_i} (N_{t_{i+1}} - N_{t_i}) - M_{t_{i+1}} N_{t_i} \middle| \mathcal{F}_{t_i} \right]$$

$$E\left[M_{t_{i+1}} N_{t_{i+1}} \middle| \mathcal{F}_{t_i} \right] = E\left[\langle M, N \rangle_{t_{i+1}} \middle| \mathcal{F}_{t_i} \right]$$

$$E\left[M_{t_i} (N_{t_{i+1}} - N_{t_i}) \middle| \mathcal{F}_{t_i} \right] = 0$$

$$E\left[M_{t_{i+1}} N_{t_i} \middle| \mathcal{F}_{t_i} \right] = M_{t_i} N_{t_i}$$

$$E\left[(M_{t_{i+1}}-M_{t_i})(N_{t_{i+1}}-N_{t_i}) \mid \mathcal{F}_{t_i}\right] = E\left[\langle M,N\rangle_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] - E\left[M_{t_i}N_{t_i} \mid \mathcal{F}_{t_i}\right]$$

$$= E\left[\langle M,N\rangle_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] - E\left[\langle M,N\rangle_{t_i} \mid \mathcal{F}_{t_i}\right] =$$

$$\overset{s.J}{=} E\left[\langle M,N\rangle_{t_{i+1}} - \langle M,N\rangle_{t_i} \mid \mathcal{F}_{t_i}\right]$$

$$\circledast \quad E\left[\sum_{i=\ell}^{k-1} \xi_i \eta_i \cancel{E}\left(\langle M,N\rangle_{t_{i+1}} - \langle M,N\rangle_{t_i} \mid \cancel{\mathcal{F}_{t_i}}\right) \mid \mathcal{F}_{t_\ell}\right] =$$

$$= E\left[\int_s^t H G \, d\langle M,N\rangle \mid \mathcal{F}_s\right]$$

Standardním způsobem ověříme

$\mathcal{H}$ prostor procesů H, G takových, že $\int H dM \int G dN - \int HG \, d\langle M, N\rangle$ je martingal

$\mathcal{H}$ obsahuje 1, obsahuje indikátory predikovatelných obdélníků

je vektorový prostor, $0 \le H^n \nearrow \underbrace{H \quad 0 \le G^n \nearrow G}_{\text{omezené}} \Rightarrow HG \in \mathcal{H}$

$H^n \in \mathcal{H}$

$G^n \in \mathcal{H}$

$\Rightarrow \mathcal{H}$ obsahuje všechny $\mathcal{F}_1$-predikovatelné omezené.