Analytic combinatorics

Lecture 4

March 31, 2021

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Let $\rho \in [0, +\infty)$ and $z \in \mathbb{C}$.

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For a complex f.p.s. $A(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$, the exponential growth rate of $A(x)$, denoted $\eta(A)$, is defined as

$$\eta(A) := \limsup_{n\to\infty} \sqrt[n]{|a_n|} \in [0, +\infty].$$

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Remark: For $G \in (0, +\infty)$, $\eta(A) = G$ means that for every $\varepsilon > 0$, there are only finitely many values of n such that $|a_n| > (G + \varepsilon)^n$, but there are infinitely many values of n such that $|a_n| > (G - \varepsilon)^n$.

"a_n grows roughly like G^n"

$A(x) = 1 + x + x^2 + x^3 + \ldots \qquad \eta(A) = 1$

$B(x) = 1 + x^2 + x^4 + x^6 + \ldots \qquad \eta(B) = 1$

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Definition

The radius of convergence of $A(x) \in \mathbb{C}[[x]]$, denoted $\rho(A)$, is defined as

$$\rho(A) := \frac{1}{\eta(A)} \in [0, +\infty], \text{ with the convention } \frac{1}{0} = +\infty.$$

The f.p.s. is said to be convergent if $\rho(A) > 0$ (or equivalently $\eta(A) < +\infty$).

Radius of convergence

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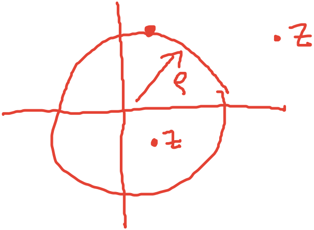
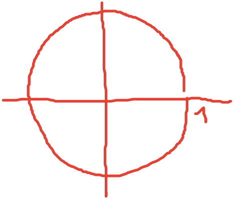
- *If* $\rho = 0$*, then for any* $z \neq 0$*, the series* $A(z)$ *does not converge.*
- *If* $\rho \in (0, +\infty)$*, then* $A(z)$ *converges for all* z *with* $|z| < \rho$ *(absolutely, locally uniformly on* $\mathcal{N}_{\cancel{\le}\rho}(0)$*), and does not converge for any* z *with* $|z| > \rho$*.*

$<\rho$



$A(z) = 1 + z + z^2 + z^3 + \ldots$

$\rho = 1$



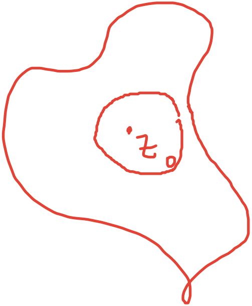
Analytic functions

Definition

Let $z_0 \in \mathbb{C}$, let f be a complex-valued function defined on an open set $\Omega \subseteq \mathbb{C}$ containing z_0. We say that f is analytic in z_0 if there is an $\varepsilon > 0$ and a power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ with $\rho(A) \geq \varepsilon$ such that for every $z \in \mathcal{N}_{<\varepsilon}(z_0)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The expression $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is then the (power) series expansion of f around the center z_0.



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Observation

Let $z_0 \in \mathbb{C}$, let $f, g \colon \mathbb{C} \to \mathbb{C}$ be two functions satisfying $f(z) = g(z + z_0)$ for all $z \in \mathbb{C}$. Then f is analytic in 0 with series expansion $\sum_{n=0}^{\infty} a_n z^n$ if and only if g is analytic in z_0 with series expansion $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Properties of analytic functions

Let f be analytic in 0 with series expansion $A(z) = \sum_{n=0}^{\infty} a_n z^n$, let g be analytic in 0 with series expansion $B(z) = \sum_{n=0}^{\infty} b_n z^n$. Then

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operations in $\mathbb{C}[[x]]$

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→ in $\mathbb{C}[[x]]$

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function composition

B composable

composition in $\mathbb{C}[[x]]$

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Consequence: convergent series form a subring of $\mathbb{C}[[x]]$.

$\rho(A) > 0$

Derivative

Definition

Let $z_0 \in \mathbb{C}$, and let f be a complex-valued function defined on an open set $\Omega \subseteq \mathbb{C}$ containing z_0. The derivative of f in z_0, denoted $f'(z_0)$, is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

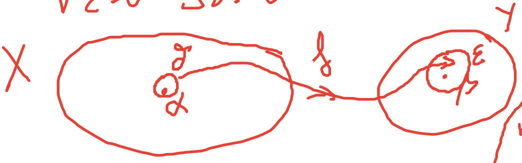
Limit: Suppose (X, d_X) and (Y, d_Y) are metric spaces, $f : X \to Y$, $\alpha \in X$, $\beta \in Y$

"$\lim_{x \to \alpha} f(x) = \beta$" means

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in X :$ $0 < d_X(x, \alpha) < \delta$

$\Downarrow$

$d_Y(f(x), \beta) < \varepsilon$



metric on $\mathbb{C}$: $|x - y|$

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- f' is defined on a neighborhood of 0, is analytic in 0, with series expansion $\frac{\mathrm{d}}{\mathrm{d}z} A(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$.

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- Consequently, f is continuous and has continuous derivatives of all orders,
- $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2a_2$, and in general $f^{(n)}(0) = n! a_n$, where $f^{(n)}$ is the derivative of f of order n.

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- In particular, the series expansion of an analytic function is unique.
- Suppose $f(0) = 0$ and $f'(0) \neq 0$ (equivalently, $a_0 = 0$ and $a_1 \neq 0$). Then there is $\varepsilon > 0$ such that f maps $\mathcal{N}_{<\varepsilon}(0)$ bijectively to an open set $\Omega \subseteq \mathbb{C}$ containing 0, and the inverse function $f^{\langle -1 \rangle} : \Omega \to \mathcal{N}_{<\varepsilon}(0)$ is analytic in 0 with series expansion $A^{\langle -1 \rangle}(z)$.

Local analytic uniqueness

Proposition

Let f be analytic in z_0. Then one of the following possibilities holds:

- *There is an $\varepsilon > 0$ such that for every $z \in \mathcal{N}_{<\varepsilon}(z_0)$, $f(z) = f(z_0)$.*
- *There is an $\varepsilon > 0$ such that for every $z \in \mathcal{N}^*_{<\varepsilon}(z_0)$, $f(z) \neq f(z_0)$.*

Pf: take $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for $z \in \mathcal{N}_{<\delta}(0)$. Distinguish 2 cases:

1) $a_n = 0$ for all $n \geq 1$: $f(z) = a_0 \quad \forall z \in \mathcal{N}_{<\delta}(0)$

2) $\exists n \geq 1$: $a_n \neq 0$. Choose smallest such n:

$$f(z) = a_0 + a_n z^n + a_{n+1} z^{n+1} + \dots$$

$$= \underbrace{a_0}_{=f(0)} + z^n \underbrace{\big(\overset{\neq 0}{a_n} + a_{n+1} z + a_{n+2} z^2 + \dots \big)}_{\text{analytic, continuous, } \Rightarrow \neq 0 \text{ on } \mathcal{N}_{<\varepsilon}(0) \text{ for some } \varepsilon > 0}$$

$\neq 0$ on $\mathcal{N}^*_{<\varepsilon}(0)$

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Corollary

Let f and g be functions analytic in z_0, with $f(z_0) = g(z_0)$. Then one of the following possibilities holds:

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- *There is an $\varepsilon > 0$ such that for every $z \in \mathcal{N}^*_{<\varepsilon}(z_0)$, $f(z) \neq g(z)$.*

Pf: Apply Proposition to $h(z) := f(z) - g(z)$

$h(z_0) = 0$ $\square$

Local analytic uniqueness

Proposition

Let f be analytic in z_0. Then one of the following possibilities holds:

- *There is an $\varepsilon > 0$ such that for every $z \in \mathcal{N}_{<\varepsilon}(z_0)$, $f(z) = f(z_0)$.*
- *There is an $\varepsilon > 0$ such that for every $z \in \mathcal{N}^*_{<\varepsilon}(z_0)$, $f(z) \neq f(z_0)$.*

Corollary

Let f and g be functions analytic in z_0, with $f(z_0) = g(z_0)$. Then one of the following possibilities holds:

- *There is an $\varepsilon > 0$ such that for every $z \in \mathcal{N}_{<\varepsilon}(z_0)$, $f(z) = g(z)$.*
- *There is an $\varepsilon > 0$ such that for every $z \in \mathcal{N}^*_{<\varepsilon}(z_0)$, $f(z) \neq g(z)$.*

Corollary

Let f and g be functions analytic in z_0, such that for every $\delta > 0$ there is a $z \in \mathcal{N}^*_{<\delta}(z_0)$ such that $f(z) = g(z)$. Then, for some $\varepsilon > 0$, we have $f(z) = g(z)$ for every $z \in \mathcal{N}_{<\varepsilon}(z_0)$.

Pf: f, g analytic $\Rightarrow$ f, g continuous $\Rightarrow$ $f(z_0) = g(z_0)$, $\square$

Examples of analytic functions

→ constant : analytic ~~an~~ in any $z_0 \in \mathbb{C}$

→ polynomial : ———— $\shortparallel$ ————

$p(x) = a_0 + a_1 x + a_2 x^2$

analytic in $z_0 \in \mathbb{C}$:

$$p(x) = a_0 + a_1 (x - z_0 + z_0) + a_2 (x - z_0 + z_0)^2$$

$$= a_0 + a_1 (x - z_0) + a_1 z_0 + a_2 \left((x-z_0)^2 + 2(x-z_0) \cdot z_0 + z_0^2 \right)$$

$$= \underbrace{(a_0 + a_1 z_0 + a_2 z_0^2)} + \underbrace{(a_1 + 2a_2)} (x - z_0) + \underbrace{a_2} (x - z_0)^2$$

$\frac{1}{p(x)} = f(x)$

p polynomial

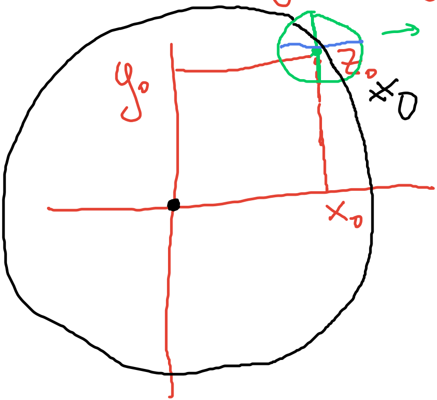
f analytic in any $z_0 \in \mathbb{C}$ s.t. $p(z_0) \neq 0$.

Examples of non-analytic functions

The functions $f_1(z) = \Re(z)$, $f_2(z) = \Im(z)$, $f_3(z) = |z|$ and $f_4(z) = \bar{z}$ are not analytic in any point.

$f_1(z)$ is not analytic in any point

$z_0 = x_0 + i y_0, \quad x_0, y_0 \in \mathbb{R}$



→ contradicts Proposition "Local uniqueness"

Let $\Omega \subseteq \mathbb{C}$ be an open set. We say that f is analytic on Ω, if f is analytic in every point of Ω.

Proposition

Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $\rho > 0$. Define a function $f : \mathcal{N}_{<\rho}(0) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then f is analytic on $\mathcal{N}_{<\rho}(0)$. Moreover, for $z_0 \in \mathcal{N}_{<\rho}(0)$, the series expansion of f with center z_0 has radius of convergence at least $\rho - |z_0|$.