1. Show that

a) $\mu \mapsto \mu(B)$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $([0,\infty], \mathcal{B}([0,\infty]))$ for every $B \in \mathcal{B}(E)$,

b) $\mu \mapsto \mu|_B$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $(\mathcal{M}, \mathfrak{M})$ for every $B \in \mathcal{B}(E)$.

c) $\mu \mapsto \int_E f(x)\,\mu(\mathrm{d}x)$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $([0,\infty], \mathcal{B}([0,\infty]))$ for every non-negative measurable function f on E.

$(E, \mathcal{B}(E))$ ... imagine $(\mathbb{R}^d, \mathcal{B}^d)$

$\mathcal{M}$ ... measures on $(E, \mathcal{B}(E))$ ... locally finite {on compac

$\mathfrak{M}$ ... σ-algebra on $\mathcal{M}$ ... „$\forall$ projections are measurable"

$B \in \mathcal{B}(E)$: $\hat{\pi}_B : \mathcal{M} \to [0,\infty]$ ... $\hat{\pi}_B(\mu) = \mu(B)$ ... projection

$\hat{\pi}_B(\Psi) = \Psi(B)$ ... we want these to be random varia

$\mathfrak{M}$ generated by $\mathcal{M}_{B,r} = \{\mu \in \mathcal{M} : \mu(B) < r\}$, $B \in \mathcal{B}$, $r \in [0,\infty$

a) $A_B : \mu \mapsto \mu(B)$ .... $(\mathcal{M}, \mathfrak{M}) \to ([0,\infty], \mathcal{B}([0,\infty]))$

$A_B = \hat{\pi}_B$ ---> $\hat{\pi}_B^{-1}([0,r)) = \{\mu \in \mathcal{M} : \mu(B) \in [0,r)\} =$

enough to check for these sets,
they generate $\mathcal{B}([0,\infty])$

b) $\mu|_B(A) = \mu(A \cap B)$, $\forall A \in \mathcal{B}(E)$

$A_B : \mu \mapsto \mu|_B$ ... $(\mathcal{M}, \mathfrak{M}) \to (\mathcal{M}, \mathfrak{M})$

take $\mathcal{M}_{B_0,r} \in \mathfrak{M}$, $A_B^{-1}(\mathcal{M}_{B_0,r}) \in \mathfrak{M}$

$$A_B^{-1}(\mathcal{M}_{B_0,r}) = \{\mu \in \mathcal{M} : \mu|_B \in \mathcal{M}_{B_0,r}\} =$$

$$= \{\mu \in \mathcal{M} : \mu(\overbrace{B \cap B_0}^{\in \mathcal{B}(E)}) < r\} = \mathcal{M}_{B \cap B_0, r} \in \mathfrak{M}$$ ←

$\{$ $\mu|_B(B_0) < r\}$

↳ one of generator

c) $f \geq 0$ measurable function on $(E, \mathcal{B}(E))$

$$A_f : \mu \longmapsto \int_E f(x)\, d\mu(x) \quad \dots \quad (\mathcal{M}, \mathfrak{M}) \to ([0,\infty], \mathcal{B}([0,\infty]))$$

- $f(x) = \mathbb{1}_B(x) = \begin{cases} 1 & \dots\ x \in B \\ 0 & \dots\ x \notin B \end{cases}$ $\qquad \int_E f(x)\, d\mu(x) = \mu(B)$

  $\Rightarrow A_f = \widetilde{\pi}_B$ ... measurable ... see a)

- $f = c \cdot \mathbb{1}_B$ ... multiple of a measurable function ✓
- $f = \sum c_i \mathbb{1}_{B_i}$ ... simple function ... sum of ▽ ✓
- $f \geq 0$ ... limit of simple functions: $f_n \nearrow f$ monoto
  (poinz

  $A_{f_n} \overset{?}{\nearrow} A_f$

  (Lévy monotone convergence theo

  $$A_{f_n} : \mu \longmapsto \int_E f_n(x)\, d\mu(x) \longrightarrow \int_E f(x)\, d\mu(x) = A_f(\mu)$$

  $\Rightarrow A_f$ is a pointwise limit of measurable functions A_{f_n} $\Rightarrow A_f$ is measurable

  provided the target space is metric with Borel σ-a

$$(\Omega, \mathcal{A}, P) \xrightarrow{\Psi} (\mathcal{M}, \mathfrak{M}) \xrightarrow{A_f} ([0,\infty], \mathcal{B}([0,\infty]))$$

$\Psi \circ A_f$ is measurable $(\Omega, \mathcal{A}, P) \longrightarrow ([0,\infty], \mathcal{B}([0,\infty]))$

$A_f(\Psi)$