

1. Show that

a) $\mu \mapsto \mu(B)$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $([0, \infty], \mathcal{B}([0, \infty]))$ for every $B \in \mathcal{B}(E)$,

b) $\mu \mapsto \mu|_B$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $(\mathcal{M}, \mathfrak{M})$ for every $B \in \mathcal{B}(E)$.

c) $\mu \mapsto \int_E f(x) \mu(dx)$ is a measurable mapping from $(\mathcal{M}, \mathfrak{M})$ to $([0, \infty], \mathcal{B}([0, \infty]))$ for every non-negative measurable function f on E .

$(E, \mathcal{B}(E))$... imagine $(\mathbb{R}^d, \mathcal{B}^d)$

μ ... measures on $(E, \mathcal{B}(E))$... locally finite { on compact

\mathfrak{M} ... σ -algebra on \mathcal{M} ... "projections are measurable"

$\mathcal{B}(\mathcal{B}(E))$: $\hat{\pi}_B : \mathcal{M} \rightarrow [0, \infty]$... $\hat{\pi}_B(\mu) = \mu(B)$... projection

$\hat{\pi}_B(\Psi) = \Psi(B)$... we want these to be random variables

\mathfrak{M} generated by $\mathcal{M}_{B, \pi} = \{\mu \in \mathcal{M} : \mu(B) < \pi\}, B \in \mathcal{B}(E), \pi \in [0, \infty)$

a) $A_B : \mu \mapsto \mu(B)$... $(\mathcal{M}, \mathfrak{M}) \rightarrow ([0, \infty], \mathcal{B}([0, \infty]))$

$A_B = \hat{\pi}_B \implies \hat{\pi}_B^{-1}([0, \pi]) = \{\mu \in \mathcal{M} : \mu(B) \leq \pi\}$
 enough to check for these sets, they generate $\mathcal{B}([0, \infty])$

b) $\mu|_B(A) = \mu(A \cap B), \forall A \in \mathcal{B}(E)$

$A_B : \mu \mapsto \mu|_B$... $(\mathcal{M}, \mathfrak{M}) \rightarrow (\mathcal{M}, \mathfrak{M})$

take $\mathcal{M}_{B_0, \pi} \in \mathfrak{M}, A_B^{-1}(\mathcal{M}_{B_0, \pi}) \in \mathfrak{M}$

$A_B^{-1}(\mathcal{M}_{B_0, \pi}) = \{\mu \in \mathcal{M} : \mu|_B \in \mathcal{M}_{B_0, \pi}\} =$
 $= \{\mu \in \mathcal{M} : \mu(\underbrace{B \cap B_0}_{\in \mathcal{B}(E)}) < \pi, \mu|_B(B_0^c) < \pi\} = \mathcal{M}_{B \cap B_0, \pi} \in \mathfrak{M}$
 { } \hookrightarrow one of generator

c) $f \geq 0$ measurable function on $(E, \mathcal{B}(E))$

$$A_f : \mu \mapsto \int_E f(x) d\mu(x) \quad \dots \quad (\mathcal{M}, \mathcal{M}) \rightarrow ([0, \infty], \mathcal{B}([0, \infty]))$$

$$\bullet \quad f(x) = \mathbb{1}_B(x) = \begin{cases} 1 & \dots x \in B \\ 0 & \dots x \notin B \end{cases} \quad \int_E f(x) d\mu(x) = \mu(B)$$

$$\Rightarrow A_f = \widetilde{\mu}_B \quad \dots \text{measurable} \quad \dots \text{see a)}$$

$\bullet \quad f = c \cdot \mathbb{1}_B \quad \dots$ multiple of a measurable function \checkmark

$\bullet \quad f = \sum c_i \mathbb{1}_{B_i} \quad \dots$ simple function \dots sum of $\mathbb{1}$ \checkmark

$\bullet \quad f \geq 0 \quad \dots$ limit of simple functions: $f_n \nearrow f$ monotone (pointwise)

$$A_{f_n} \xrightarrow{?} A_f$$

(Levy monotone convergence theorem)

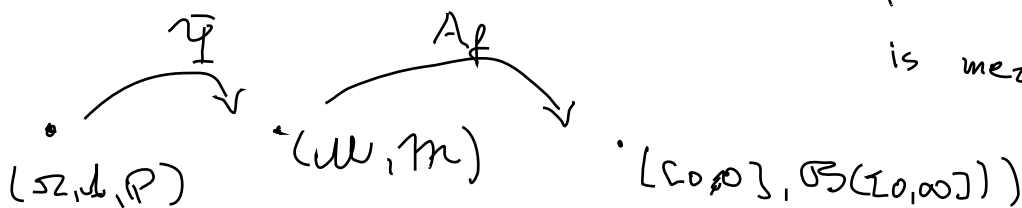
$$A_{f_n} : \mu \mapsto \int_E f_n(x) d\mu(x) \rightarrow \int_E f(x) d\mu(x) = A_f(\mu)$$

$\Rightarrow A_{f_n}$ is a pointwise limit of measurable

functions $A_{f_n} \Rightarrow A_f$ is measurable

\uparrow provided the target space

is metric with Borel σ -



$\Psi = A_f$ is measurable $(\mathcal{S}, \mathcal{B}, \mathcal{P}) \rightarrow ([0, \infty], \mathcal{B}([0, \infty]))$

$$A_f(\Psi)$$