NMAI059 Probability and statistics 1
Class 5

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Overview

Random vectors

Conditional distribution

Continuous random variables

Basic description of random vectors

- X, Y – random variables on the same probability space $(\Omega, \mathcal{F}, P)$.
- We wish to treat (X, Y) as one object – a random vector.
- How to do that?
- Example: we roll twice a 4-sided dice, $X =$ first outcome, $Y =$ second one.

Joint distribution

Definition

For a discrete r.v. X, Y on a probability space $(\Omega, \mathcal{F}, P)$ we define their joint PMF $p_{X,Y} : \mathbb{R}^2 \to [0,1]$ by a formula

$$p_{X,Y}(x,y) = P(\{\omega \in \Omega : X(\omega) = x \& Y(\omega) = y\}).$$

For this we need that for each $x, y \in \mathbb{R}$ we have $\{\omega \in \Omega : X(\omega) = x \& Y(\omega) = y\} \in \mathcal{F}$, otherwise we do not consider (X,Y) as a random vector.

- We can define it also for more than two r.v.'s $p_{X_1,\dots,X_n}(x_1,\dots,x_n)$.

Marginal distribution

- Given $p_{X,Y}$, how to find the distribution of each of the coordinates, that is p_X and p_Y?

Independence of r.v.'s

Definition

Discrete r.v.'s X, Y are independent if for every $x, y \in \mathbb{R}$ the events $\{X = x\}$ a $\{Y = y\}$ are independent. That happens if and only if

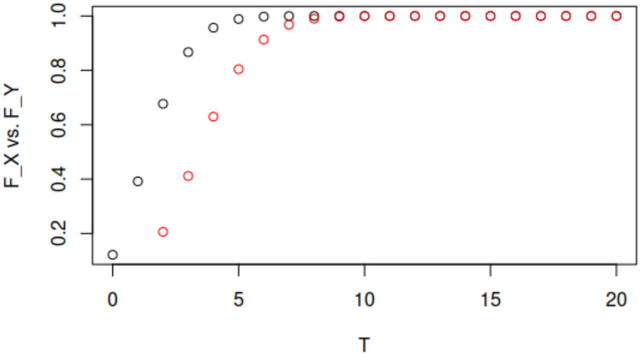
$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

Example: Multinomial distribution

- On a die we roll i with probability p_i for $i = 1, \ldots, 6$. We roll the die n-times and let X_i be the number of rolls when i came up.

Coupling – nontrivial use of joint distributions

- $X \sim Bin(n, p)$ and $Y \sim Bin(n, q)$ for $p < q$
- What can be said about F_X and F_Y?
- $\sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$ is an increasing function of p – but why?



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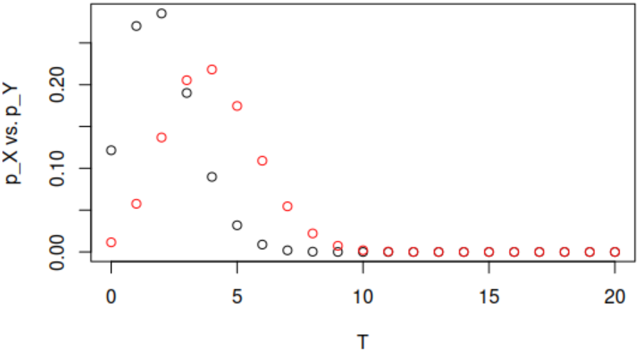

Coupling

- $X = \sum_{i=1}^{n} X_i$, where $X_1, \ldots, X_n$ are independent
- $Y = \sum_{i=1}^{n} Y_i$, where $Y_1, \ldots, Y_n$ are independent
- Joint distribution of X and Y is not determined, it can be arbitrary.
- We make it so that X and Y are not independent, more so, always $X \leq Y$.
- It suffices to define $Y_i =$

Product of independent r.v.'s

Theorem

For independent discrete r.v.'s X, Y we have

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y).$$

Function of a random vector

Theorem

Suppose X, Y are discrete r.v.'s on $(\Omega, \mathcal{F}, P)$, let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function.

- *Then $Z = g(X, Y)$ is a r.v. on $(\Omega, \mathcal{F}, P)$*
- *and it satisfies*

$$\mathbb{E}(g(X,Y)) = \sum_{x \in ImX} \sum_{y \in ImY} g(x,y) P(X = x, Y = y),$$

whenever the sum is defined.

Theorem (Linearity of expectation)

For X, Y r.v.'s (independence is not needed!) and $a, b \in \mathbb{R}$ we have

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y).$$

Sum of independent r.v.'s

- Given $p_{X,Y}$, how to find the distribution of the sum, $Z = X + Y$?

Sum of r.v.'s – convolution

Theorem (Convolution formula)

Let X, Y be discrete random variables. Then their sum $Z = X + Y$ has PMF given by

$$P(Z = z) = \sum_{x \in Im(X)} P(X = x, Y = z - x).$$

If we further assume that X, Y are independent, then

$$P(Z = z) = \sum_{x \in Im(X)} P(X = x)P(Y = z - x).$$

Example of a convolution

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Conditional PMF

X, Y – discrete random variables on $(\Omega, \mathcal{F}, P)$, $A \in \mathcal{F}$

- $p_{X|A}(x) := P(X = x \mid A)$
  example: X is outcome of a roll of a die, $A =$ we got an even number
- $p_{X|Y}(x|y) = P(X = x \mid Y = y)$ example: X, Z is an outcome of two independent die rolls, $Y = X + Z$.

$$p_{X|Y}(6|10) =$$

- $p_{X|Y}$ from $p_{X,Y}$:

Joint vs. conditional PMF

$p_{X,Y}$	...	10	11	12
1				
2				
3				
4				
5				
6				

$p_{X\|Y}$	...	10	11	12
1				
2				
3				
4				
5				
6				

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General random variable

Definition

Random variable on $(\Omega, \mathcal{F}, P)$ *is a mapping* $X : \Omega \to \mathbb{R}$*, such that for each* $x \in \mathbb{R}$

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}.$$

- discrete r.v. is a r.v.

CDF

Definition

Cumulative distribution function, CDF of a r.v. X is a function

$$F_X(x) := P(X \leq x) = P(\{\omega \in \Omega : X(\omega) \leq x).$$

- F_X is a nondecreasing function
- $\lim_{x \to -\infty} F_X(x) = 0$
- $\lim_{x \to +\infty} F_X(x) = 1$
- F_X is right-continuous

CDF examples

Quantile function

For a r.v. X we define its *quantile function* $Q_X : [0,1] \to \mathbb{R}$ by

$$Q_X(p) := \min\{x \in \mathbb{R} : p \leq F_X(x)\}$$

- If F_X is continuous, then $Q_X = F_X^{-1}$.
- $Q_X(1/2) =$ median (watch out if F_X is not strictly increasing!)
- $Q_X(10/100) =$ tenth percentile, etc.

Continuous random variable

Definition

R.v. X is called continuous, if there is nonnegative real function f_X such that

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt.$$

(Sometimes such X is said to be absolutely continuous.)
Function f_X is called the probability density function, pdf of X.

Using density

Theorem

Let X be a continuous r.v. with density f_X. Then

1. $P(X = x) = 0$ *for every* $x \in \mathbb{R}$.
2. $P(a \leq X \leq b) = \int_a^b f_X(t)dt$ *for every* $a, b \in \mathbb{R}$.

Uniform distribution

- R.v. X has a uniform distribution on $[a, b]$, we write $X \sim U(a, b)$, if $f_X(x) = 1/(b - a)$ for $x \in [a, b]$ and $f_X(x) = 0$ otherwise.

Universality of a uniform distribution

Theorem

Let F be a function "of CDF-type": nondecreasing right-continuous function with $\lim_{x \to -\infty} F(x) = 0$ a $\lim_{x \to +\infty} F(x) = 1$. Let Q be the corresponding quantile function.

1. *Let $U \sim U(0,1)$ and $X = Q(U)$. Then X has CDF F.*
2. *Let X be a r.v. with CDF $F_X = F$, suppose F is increasing. Then $F(X) \sim U(0,1)$.*