NMFM 402, Practical 4, GLM – multip. tariff struct.

Ex 1. Denote initial data: $y_{ij} = y_m$ ... claims per policy in m-th cell

$v_{ij} = v_m$ ... number of policies in m-th cell

and after change: $y_m^{(K)} = y_m$ ... ~~old~~ claims per policy unchanged

$v_m^{(K)} = K \cdot v_m$ ... nr. of policies multiplied by K

First, let's review the effect of change on point estimates. ~~Q~~ MLE of β is obtained from (2.19). After change, both sides ~~multip~~ multiply by K $\Longrightarrow$ solution (the MLE) remains unchanged. Consequently, risk factors $e^{\cancel{\beta_r}} e^{\beta_{r,l}}$ and expected claims per policy $e^{(Z\beta)_m}$ do not change as well. Hence, the change in exposure does not affect the point estimates. Let's see, how it affects their variabilities.

(a) Covariance matrix of MLE can be approximated by the inverse of Fisher information matrix.

$$\operatorname{var}(\beta_{MLE}) \approx \left[J(\hat{\beta})\right]^{-1}, \quad \text{where}$$

$$\left(J\hat{\beta}\right)_{j,r} = -E\left(\frac{\partial l}{\partial\beta_j \partial\beta_r}\right) = -E\left(\frac{\partial}{\partial\beta_r}\left[\sum_m \frac{y_m - \exp\{(Z\beta)_m\}}{\frac{1}{v_m}} \cdot z_{ml}\right]\right)$$

$$= \sum_m v_m \cdot \exp\{(Z\hat{\beta})_m\} \cdot z_{mr} \cdot z_{ml}$$

after the change, the point estimate $\hat{\beta}$ is unaffected and so

$$\left(J^{(K)}(\hat{\beta})\right)_{jr} = \sum_m K \cdot v_m \cdot \exp\{(Z\hat{\beta})_m\} \cdot z_{mr} \cdot z_{mj} = K \cdot \left[J(\hat{\beta})\right]_{jr}$$

In result $\boxed{\underline{var\left(\beta^{(K)}_{MLE}\right) \approx \frac{1}{K} \ var(\beta)}}$.

In particular, ~~variances~~ variance of MLE of each β_j is K-times lower after K-times increase in nr. of policies.

(b) For variance of risk factors e^{β_j}, we can use linear approximation of exp. function by Taylor polynomial at point estimate $\hat{\beta}_j$:

$$e^{\beta_{MLE,j}} \approx e^{\hat{\beta}_j} + e^{\hat{\beta}_j} \cdot \left(\beta_{MLE,j} - \hat{\beta}_j\right)$$

the estimator as random variable

point estimate, fixed ~~valu~~ value

This implies $var\left(e^{\beta_{MLE,j}}\right) \approx \left(e^{\hat{\beta}_j}\right)^2 \cdot var\left(\beta_{MLE,j}\right)$,

and $\boxed{\underline{var\left(e^{\beta^{(K)}_{MLE,j}}\right) \approx \frac{1}{K} \cdot var\left(e^{\beta_{MLE,j}}\right)}}$

(c) $\widehat{EY}_m = e^{(Z\beta_{MLE})_m}$

$$var\left[(Z\beta_{MLE})_m\right] = var\left(\underset{\substack{\uparrow \\ m\text{-th row of } Z}}{Z_{m\cdot}}\beta_{MLE}\right) = Z_{m\cdot} \ var(\beta_{MLE}) \cdot Z^T_{m\cdot}$$

$$\Rightarrow var\left[\left(Z\beta^{(K)}_{MLE}\right)_m\right] = Z_{m\cdot} \ var\left(\beta^{(K)}_{MLE}\right) \cdot Z^T_{m\cdot} \approx \frac{1}{K} Z_{m\cdot} \ var(\beta_{MLE}) \cdot Z^T_{m\cdot} = \frac{1}{K} \ var\left[\left(Z\beta^{(K)}_{MLE}\right)_m\right]$$

Similarly to (b), we conclude

$$\underline{\underline{var\left(\widehat{\mathbb{E}Y_m}^{(k)}\right)}} = var\left(e^{\left(Z\beta_{MLE}^{(k)}\right)_m}\right) \approx \cancel{var}\ \frac{1}{K} \cdot var\left(e^{\left(Z\hat{\beta}_{MLE}\right)_m}\right) =$$

$$\underline{\underline{= \frac{1}{K}\, var\left(\mathbb{E}\hat{Y}_m\right)}}$$

After the change, the variance of the estimated expected nr. of claims per policy decreased K-times.
approximately

To conclude, increasing number of policies K-times did not change point estimates, but it ~~incre~~ decreased variances of the estimator approx. K-times.