Analytic combinatorics

Lecture 3

March 24, 2021

Labelled classes

Definition

A labelled combinatorial class is a set $\mathcal{A}$ in which every object $\alpha \in \mathcal{A}$ has a vertex set (or ground set or set of labels), denoted $V(\alpha)$, which is a finite subset of $\mathbb{N}$, satisfying the following conditions:

- For every finite set $X \subseteq \mathbb{N}$, there are only finitely many objects $\alpha \in \mathcal{A}$ with $V(\alpha) = X$.
- For every two finite sets $X, Y \subseteq \mathbb{N}$ of the same size, the number of objects in $\mathcal{A}$ with vertex set X is the same as the number of those with vertex set Y.

Examples: graphs, permulations,

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- $\mathcal{A}_*$ denotes the set $\bigcup_{n=0}^{\infty} \mathcal{A}_n$ of all the normalized elements of $\mathcal{A}$.

Exponential generating functions

Definition

Let $\mathcal{A}$ be a labelled combinatorial class, let $a_n = |\mathcal{A}_n|$. The exponential generating function of $\mathcal{A}$, denoted EGF($\mathcal{A}$) is the f.p.s.

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}. \quad = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

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Remark: We may also write

$$\text{EGF}(\mathcal{A}) = \sum_{\alpha \in \mathcal{A}_*} \frac{x^{|\alpha|}}{|\alpha|!}.$$

Operations with labelled classes and EGFs

Observation

If $\mathcal{A}$ and $\mathcal{B}$ are disjoint labelled comb. classes, then $\mathsf{EGF}(\mathcal{A} \cup \mathcal{B}) = \mathsf{EGF}(\mathcal{A}) + \mathsf{EGF}(\mathcal{B})$.

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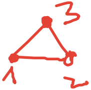
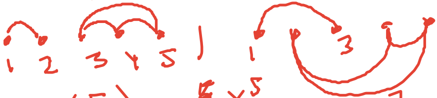
Let $\mathcal{A}$ and $\mathcal{B}$ be labelled comb. classes. Their labelled product, denoted $\mathcal{A} \otimes \mathcal{B}$, is the labelled comb. class

$$\{(\alpha, \beta);\ \alpha \in \mathcal{A} \ \&\ \beta \in \mathcal{B} \ \&\ \underline{V(\alpha) \cap V(\beta) = \emptyset}\},$$

with $V((\alpha, \beta)) = V(\alpha) \cup V(\beta)$.

Ex: $\mathcal{A}_* = \{$  $\}$ EGF: $\frac{x^2}{2!}$

$\mathcal{B}_* = \{$  $\}$ EGF: $\frac{x^3}{3!}$

$(\mathcal{A} \otimes \mathcal{B})_* = \{$  $, \dots$

$\binom{5}{2}$ EGF: $\binom{5}{2} \cdot \frac{\cancel{5}\, x^5}{5!} = \frac{x^2}{2!} \cdot \frac{x^3}{3!}$

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Proof.

$$[x^n]\,\mathrm{EGF}(\mathcal{A} \otimes \mathcal{B}) = \frac{|(\mathcal{A} \otimes \mathcal{B})_n|}{n!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} |\mathcal{A}_k| \cdot |\mathcal{B}_{n-k}| = \sum_{k=0}^{n} \frac{|\mathcal{A}_k|}{k!} \cdot \frac{|\mathcal{B}_{n-k}|}{(n-k)!}$$

$$= \sum_{k=0}^{n} ([x^k]\,\mathrm{EGF}(\mathcal{A})) \cdot ([x^{n-k}]\,\mathrm{EGF}(\mathcal{B})) = [x^n]\,\mathrm{EGF}(\mathcal{A})\,\mathrm{EGF}(\mathcal{B}). \quad \square$$

Some more operations

Let $\mathcal{A}$ be a labelled comb. class, let $A(x)$ be its EGF.

- $\mathcal{A}^{\otimes \mathbf{2}} = \mathcal{A} \otimes \mathcal{A}$ is the class of ordered pairs of vertex-disjoint objects from $\mathcal{A}$. Its EGF is $A(x)^2$.

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- Assume $\mathcal{A}_0 = \emptyset$. Then $\{\emptyset\} \cup \mathcal{A} \cup \mathcal{A}^{\otimes 2} \cup \mathcal{A}^{\otimes 3} \cup \cdots$ is the class of ordered sequences of vertex-disjoint objects from $\mathcal{A}$. Its EGF is

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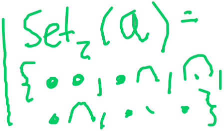
- Assume $\mathcal{A}_0 = \emptyset$, and fix $k \in \mathbb{N}_0$. Let $\mathrm{Set}_k(\mathcal{A})$ be the labelled comb. class of all the k-element sets $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ where the α_i are vertex-disjoint objects from $\mathcal{A}$, and $V(\{\alpha_1, \alpha_2, \ldots, \alpha_k\}) = V(\alpha_1) \cup V(\alpha_2) \cup \cdots \cup V(\alpha_k)$.



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- Assume $\mathcal{A}_0 = \emptyset$. Define $\mathrm{Set}(\mathcal{A}) = \bigcup_{k=0}^{\infty} \mathrm{Set}_k(\mathcal{A})$. Then

$$\mathrm{EGF}(\mathrm{Set}(\mathcal{A})) = 1 + A(x) + \frac{A(x)^2}{2!} + \frac{A(x)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = \exp(A(x)),$$

where $\exp(x)$ (or e^x) denotes the f.p.s. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

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$$\frac{1}{k!} C^k(x) = \text{EGF}(\text{graphs with } k \text{ comp.})$$

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- Hence $C(x) = L(G(x) - 1)$.
- So $c_n = n![x^n]L(G(x) - 1)$, which can be evaluated in time polynomial in n.

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set partitions of $\{1,2,3\}$:

$\{\{1,2,3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$,
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- $\mathcal{P} \cong \mathrm{Set}(\mathcal{B})$, hence

$$\mathrm{EGF}(\mathcal{P}) = \sum_{k=0}^{\infty} \frac{(\exp(x) - 1)^k}{k!} = \exp(\exp(x) - 1).$$

Weighted classes

Definition

A weighted labelled combinatorial class is a pair $(\mathcal{A}, w)$ where $\mathcal{A}$ is a labelled comb. class, and $w\colon \mathcal{A} \to K$ is a function such that for any two finite sets $X, Y \subseteq \mathbb{N}$ of the same cardinality, there is a weight-preserving bijection between objects on vertex set X and objects on vertex set Y.

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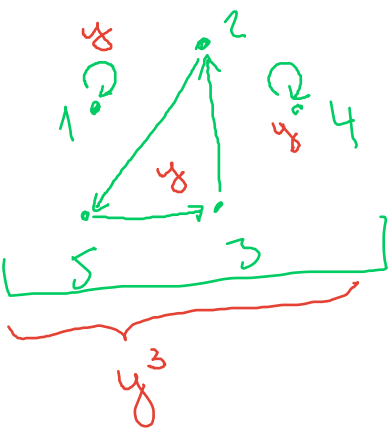
- $\mathrm{EGF}((\mathcal{A}, w_{\mathcal{A}}) \otimes (\mathcal{B}, w_{\mathcal{B}})) = \mathrm{EGF}(\mathcal{A}, w_{\mathcal{A}})\, \mathrm{EGF}(\mathcal{B}, w_{\mathcal{B}})$.

Example: cycles and permutations

Question: What is the expected number of cycles in a random permutation of $[n]$?



$1 \to 1$

$2 \to 5$

$5 \to 3$

$3 \to 2$

$4 \to 4$

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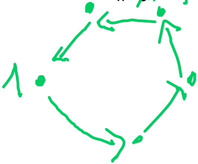

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- Note also, that permutations with exactly k cycles correspond to $\mathrm{Set}_k(\mathcal{C})$ and have EGF $\frac{1}{k!}C(x)^k$, while $\mathcal{P}$ corresponds to $\mathrm{Set}(\mathcal{C})$ and hence

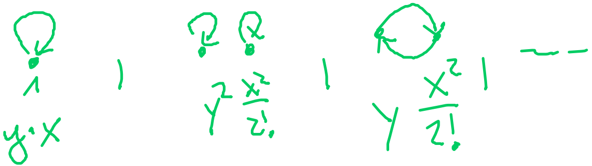
$$\mathrm{EGF}(\mathcal{P}) = \exp(C(x)) = \frac{1}{1-x}.$$

Example continued

Question: What is the expected number of cycles in a random permutation of $[n]$?

To answer the question, follow these steps:

1. To a permutation $\pi \in \mathcal{P}$ assign the weight $w(\pi) = y^{c(\pi)}$, where $c(\pi)$ is the number of cycles of π and y is a new formal variable.



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3. Calculate

$$D(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} P(x,y) = \sum_{n,k} p_{n,k} k y^{k-1} \frac{x^n}{n!}$$

$$\begin{aligned} D(x,1) &= \sum_{n,k} p_{n,k} k \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} (\text{total number of cycles in permutations of } [n]) \frac{x^n}{n!} \end{aligned}$$

$$\begin{aligned} [x^n] D(x,1) &= \frac{\text{total number of cycles in permutations of } [n]}{n!} \\ &= \text{expected number of cycles in a random permutation} \end{aligned}$$

Example finished

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