3. Consider a spherical model for the autocovariance function of a stationary isotropic random field:

$$C(\|h\|) = \sigma^2 \frac{|b(o,\varrho) \cap b(h,\varrho)|}{|b(o,\varrho)|}, \ h \in \mathbb{R}^d.$$

This model is valid in the dimension d and all the lower dimensions, see Exercise 1 above. However, it is not valid in higher dimensions. Express this autocovariance function for $d = 1$ and check that it is a positive semidefinite function. Show that this function considered in $\mathbb{R}^2$ (using $\|h\|, h \in \mathbb{R}^2$, as its argument) is not positive semidefinite.

Hint: Consider the points $x_{ij} = (i\sqrt{2}\varrho, j\sqrt{2}\varrho)$, $i, j = 1, \ldots, 8$ and the coefficients $\alpha_{ij} = (-1)^{i+j}$.

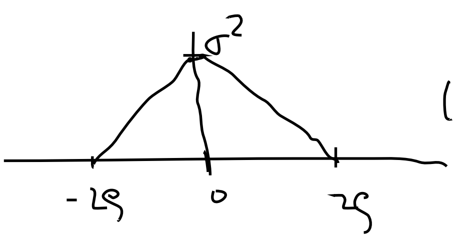
$d = 1$ ... $b(o,\varrho) = (-\varrho, \varrho)$, $b(h,\varrho) = (h-\varrho, h+\varrho)$

$h \in \mathbb{R}$ $\quad |b(o,\varrho)| = 2\varrho$

$|b(o,\varrho) \cap b(h,\varrho)| = 2\varrho - |h|, \ |h| \le 2\varrho$

$\to C(\|h\|) = \sigma^2 \cdot \left(1 - \frac{\|h\|}{2\varrho}\right)^+, \ h \in \mathbb{R} \quad = 0, \ |h| > 2\varrho$

$\infty > \sigma^2 > 0$



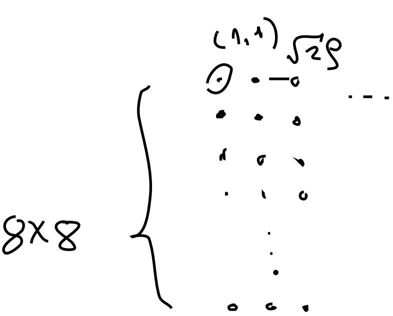
„tent model"

↳ positive semidefinite function:

$(1-\|h\|)^+$ char. function of:

$$f(x) = \frac{1}{\pi} \frac{1-\cos x}{x^2}, \ x \in \mathbb{R}$$

now choose $d = 2$, $C(\|h\|), h \in \mathbb{R}^2$, is NOT PSD



if $C(\|h\|), h \in \mathbb{R}^2$, would be PSD,

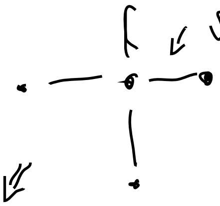
then it must hold:

$$(*) = \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \alpha_{ij} \alpha_{kl} C(\|x_{ij} - x_{kl}\|) \ge 0$$

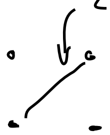
$\forall n, \forall \alpha_{ij}, \forall x_{ij}$

choose $\alpha_{ij} = (-1)^{i+j}$ $\quad i,j = 1,\ldots,8$

$x_{ij} = (i\sqrt{2}\varrho, j\sqrt{2}\varrho)$

non-zero contributions from:  $\sqrt{2}\rho < 2\rho$

 $0 < 2\rho$

zero:  $2\rho \Rightarrow C(2\rho) = 0$

$2\sqrt{2}\rho > 2\rho \Rightarrow C(\,) = 0$

$C(\sqrt{2}\rho) = \sigma^2\left(1 - \frac{\sqrt{2}}{2}\right)$

$\alpha_{ij}\alpha_{kl} = -1$

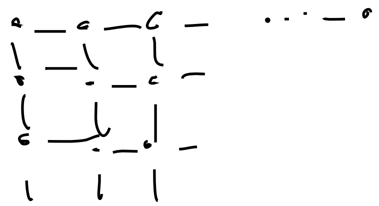
$2 \times 112 = 224$

$x_{ij} = x_{kl} \Rightarrow i = k, \; j = l$

$C(0) = \sigma^2$

$\alpha_{ij}\alpha_{kl} = 1$

$$(*) = \sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \alpha_{ij}\alpha_{kl}\, C(\|x_{ij} - x_{kl}\|) = \underbrace{64}_{64\times} \cdot 1 \cdot C(0) - 224\, C(\sqrt{2}\rho)$$



7 edges

7

⋮

7

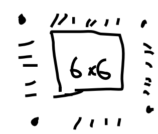
$8 \times 7 = 56$ horizontal edges

$+ 56$ vertical edges

36 points with 4 neighbours

24 3

4 2

 $\to$

$$(**) = 64\sigma^2 - 224\sigma^2\left(1 - \frac{\sqrt{2}}{2}\right) = \sigma^2\left(64 - 224 + 112\cdot\sqrt{2}\right) =$$

$$= \sigma^2\left(112\sqrt{2} - 160\right) < 0$$

$28\sqrt{2} - 40$

$7\sqrt{2} - 10 \quad /^2$

$49 \cdot 2 < 100$