5. Determine the spectral density of a weakly stationary random field with the autocovariance function

$$(*) = C(h) = \exp\{-\|h\|^2\},\ h \in \mathbb{R}^d.$$

Theorem: weakly stat. L_2-continuous RF:

$$C(h) = \int_{\mathbb{R}^d} e^{i\omega^T h}\, dS(\omega),\ h \in \mathbb{R}^d$$

$$\longrightarrow C(h) = \int_{\mathbb{R}^d} e^{i\omega^T h} s(\omega)\, d\omega \quad \text{... if spekt. density exists}$$

if $\int_{\mathbb{R}^d} |C(h)|\, dh < \infty \Rightarrow s(\omega) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\omega^T h} C(h)\, dh,\ \omega \in \mathbb{R}^d$

for $X \sim N_d(0, \sigma^2 I_d)$ ... $E e^{it^T X} = \exp\{-\frac{\sigma^2 \|t\|^2}{2}\},\ t \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} e^{it^T x} f(x)\, dx = \exp\{-\frac{\sigma^2\|t\|^2}{2}\} = (*) = C(h)$$

$\omega \ldots x$, $h \ldots t$ $\Big\}$ $s(\omega) \ldots f_{\sigma^2}(x)$ ... $\sigma^2 = 2$

• does $s(\omega)$ exist? already clear when we found it and checked integrability

$\int_{\mathbb{R}^d} \exp\{-\|h\|^2\}\, dh < \infty$? YES, $\exp\{-\|h\|^2\}$ is up to cons p.d.f. of $N_d(0, \frac{1}{2} I_d)$

$\Rightarrow s(\omega)$ exists.

$$f_{\sigma^2}(x) = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{-\frac{\|x\|^2}{2\sigma^2}},\ x \in \mathbb{R}^d \quad \ldots \sigma^2 = 2$$

$N_d(0, 2I_d)$. ... $\underline{s(x)} = f(x)$.