Analytic combinatorics

Lecture 2

March 17, 2021

Differentiation of formal power series

Recall: $K[[x]]$ is the ring of formal power series over a coefficient ring K.

Definition

The (formal) derivative of a f.p.s. $A(x) = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$, denoted $\frac{\mathrm{d}}{\mathrm{d}x}A(x)$, is the formal power series

$$a_1 + 2a_2x + 3a_3x^n + \cdots = \sum_{n=0}^{\infty} na_nx^{n-1}.$$

$$\underbrace{1+1+1+\dots+1}_{n \text{ copies}}$$

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Fact

The formal derivative satisfies formulas analogous to the 'analytic' derivative known from calculus:

- $\frac{\mathrm{d}}{\mathrm{d}x}(A(x) + B(x)) = \left(\frac{\mathrm{d}}{\mathrm{d}x} A(x)\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x} B(x)\right)$,
- $\frac{\mathrm{d}}{\mathrm{d}x}(A(x)B(x)) = \left(\frac{\mathrm{d}}{\mathrm{d}x} A(x)\right) B(x) + A(x) \left(\frac{\mathrm{d}}{\mathrm{d}x} B(x)\right)$,
- $\frac{\mathrm{d}}{\mathrm{d}x}(A(B(x)) = \left(\frac{\mathrm{d}}{\mathrm{d}x} B(x)\right) A'(B(x))$, *where* $A'(x) = \frac{\mathrm{d}}{\mathrm{d}x} A(x)$.
- *etc.*

Combinatorial classes

Definition

A combinatorial class is a set $\mathcal{A}$ whose every element $\alpha \in \mathcal{A}$ has an associated size, denoted $|\alpha|$, such that these properties hold:

- $|\alpha|$ is a non-negative integer for every $\alpha \in \mathcal{A}$, and
- for every $n \in \mathbb{N}_0$ there are only finitely many $\alpha \in \mathcal{A}$ such that $|\alpha| = n$.

Notation:

- $\mathcal{A}_n$ . . . the set of elements of $\mathcal{A}$ of size n
- a_n . . . the cardinality of $\mathcal{A}_n$
- $\mathcal{A} \cong \mathcal{B}$ means $|\mathcal{A}_n| = |\mathcal{B}_n|$ for each $n \in \mathbb{N}_0$.

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Observe: $\mathrm{OGF}(\mathcal{A}) = \sum_{\alpha \in \mathcal{A}} x^{|\alpha|}$.

Series and classes

Observation

Suppose $\mathcal{A}$ and $\mathcal{B}$ are disjoint combinatorial classes. Then

$$\mathsf{OGF}(\mathcal{A} \cup \mathcal{B}) = \mathsf{OGF}(\mathcal{A}) + \mathsf{OGF}(\mathcal{B}).$$

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Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes. Their Cartesian product $\mathcal{A} \times \mathcal{B}$ is the combinatorial class $\{(\alpha, \beta);\ \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$, with $|(\alpha, \beta)| = |\alpha| + |\beta|$.

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Lemma

$\mathrm{OGF}(\mathcal{A} \times \mathcal{B}) = \mathrm{OGF}(\mathcal{A})\,\mathrm{OGF}(\mathcal{B})$.

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Lemma

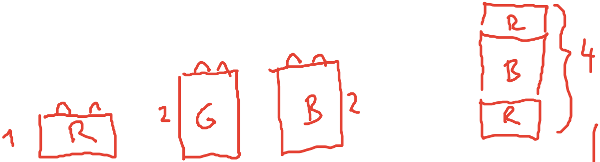
$\mathrm{OGF}(\mathcal{A} \times \mathcal{B}) = \mathrm{OGF}(\mathcal{A})\,\mathrm{OGF}(\mathcal{B})$.

Proof:

$$\begin{aligned}\mathrm{OGF}(\mathcal{A} \times \mathcal{B}) &= \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} x^{|(\alpha,\beta)|} = \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} x^{|\alpha|+|\beta|} = \sum_{\alpha\in\mathcal{A}}\sum_{\beta\in\mathcal{B}} x^{|\alpha|+|\beta|} \\ &= \left(\sum_{\alpha\in\mathcal{A}} x^{|\alpha|}\right)\left(\sum_{\beta\in\mathcal{B}} x^{|\beta|}\right) = \mathrm{OGF}(\mathcal{A})\,\mathrm{OGF}(\mathcal{B}).\end{aligned}$$

Suppose we have an unlimited number of three types of lego blocks: red blocks have height one, green blocks have height two, and blue blocks have also height two. We stack blocks on top of each other to build a tower. The 'size' of a tower is its total height. Let t_n be the number of possible towers of height n.

Goal: Formula for $\sum_{n=0}^{\infty} t_n x^n$ (and for t_n, if possible).

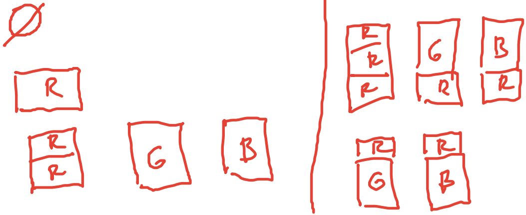


$t_3 = 5$

$t_0 = 1$

$t_1 = 1$

$t_2 = 3$



Toy example

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- $\mathcal{T}$ . . . combinatorial class of all the towers
- $T(x)$= $\mathrm{OGF}(\mathcal{T}) = \sum_{n=0}^{\infty} t_n x^n = 1 + x + 3x^2 + \cdots$

Toy example (continued)

Towers made of exactly two blocks correspond to $\mathcal{B} \times \mathcal{B}$, their OGF is therefore

$$\begin{aligned} B^2(x) &= (x + x^2 + x^2)(x + x^2 + x^2) \\ &= xx + xx^2 + xx^2 + x^2x + x^2x^2 + x^2x^2 + x^2x + x^2x^2 + x^2x^2 \\ &= x^2 + 4x^3 + 4x^4. \end{aligned}$$

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Observation:

$$\mathcal{T} = \{\emptyset\} \cup \mathcal{B} \cup (\mathcal{B} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{B} \times \mathcal{B}) \cup \cdots$$

Therefore,

$$T(x) = 1 + B(x) + B^2(x) + B^3(x) + \cdots = \frac{1}{1 - B(x)} = \frac{1}{1 - x - 2x^2}.$$

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Note: For a different set $\mathcal{B}$ of blocks (even infinite) the above argument still works, provided

- there are only finitely many blocks of each height, and
- there is no block of height 0.

Toy example (different approach)

Idea: Each tower is either empty, or consists of a bottom block and an arbitrary tower on top of it. Formally:

$$\underbrace{\mathcal{T}}_{} \cong \underbrace{\{\emptyset\}}_{} \cup \underbrace{(\mathcal{B} \times \mathcal{T})}_{\text{nonempty towers}}.$$

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Remark:

$$\frac{1}{1 - x - 2x^2} = \frac{1}{(1+x)(1-2x)} = \frac{2}{3} \cdot \frac{1}{1-2x} + \frac{1}{3} \cdot \frac{1}{1+x},$$

hence $t_n = \frac{2^{n+1}}{3} + \frac{(-1)^n}{3}$.

Adding weight

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$$\mathrm{OGF}(\mathcal{A}, w) = w_0 + w_1 x + w_2 x^2 + \cdots = \sum_{n=0}^{\infty} w_n x^n, \quad \in K[[x]]$$

where $w_n = \sum_{\alpha \in \mathcal{A}_n} w(\alpha)$.

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Observe: $\mathrm{OGF}(\mathcal{A}, w) = \sum_{\alpha \in \mathcal{A}} w(\alpha) x^{|\alpha|}$.

Operations with weighted classes

Definition

For two weighted combinatorial classes $(\mathcal{A}, w_\mathcal{A})$ and $(\mathcal{B}, w_\mathcal{B})$, with $\mathcal{A}$ and $\mathcal{B}$ disjoint, we let $(\mathcal{A}, w_\mathcal{A}) \cup (\mathcal{B}, w_\mathcal{B})$ be the weighted combinatorial class $(\mathcal{A} \cup \mathcal{B}, w_\cup)$ with $w_\cup(\alpha) = w_\mathcal{A}(\alpha)$ for $\alpha \in \mathcal{A}$, and $w_\cup(\alpha) = w_\mathcal{B}(\alpha)$ for $\alpha \in \mathcal{B}$.

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Observation

- $\mathrm{OGF}((\mathcal{A}, w_\mathcal{A}) \cup (\mathcal{B}, w_\mathcal{B})) = \mathrm{OGF}(\mathcal{A}, w_\mathcal{A}) + \mathrm{OGF}(\mathcal{B}, w_\mathcal{B})$, *and*
- $\mathrm{OGF}((\mathcal{A}, w_\mathcal{A}) \times (\mathcal{B}, w_\mathcal{B})) = \mathrm{OGF}(\mathcal{A}, w_\mathcal{A})\,\mathrm{OGF}(\mathcal{B}, w_\mathcal{B})$.

Why weights?

There are two typical situations where weight are useful:

- Weight w defines a probability distribution over a combinatorial class $\mathcal{A}$, i.e., $w(\alpha) \geq 0$ for each $\alpha \in \mathcal{A}$, and $\sum_{\alpha \in \mathcal{A}} w(\alpha) = 1$.

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Observe: If $(\mathcal{A}, w_{\mathcal{A}})$ and $(\mathcal{B}, w_{\mathcal{B}})$ are weighted comb. classes whose weights are probability distributions, and $(\mathcal{A} \times \mathcal{B}, w_{\times}) = (\mathcal{A}, w_{\mathcal{A}}) \times (\mathcal{B}, w_{\mathcal{B}})$, then $w_{\times}$ defines a probability distribution over $\mathcal{A} \times \mathcal{B}$.

$w_{\times}(\underbrace{(\alpha, \beta)}) = w_{a}(\alpha)\, w_{B}(\beta)$

Weights as probability distributions

1 2 2

Consider again red, green and blue lego blocks, with heights as before. Suppose we randomly select blocks with probabilities $w(red) = \frac{1}{2}$, $w(green) = \frac{1}{3}$ and $w(blue) = \frac{1}{6}$.

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Then

$$
\begin{aligned}
B_w^2(x) &= (\frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{6}x^2)(\frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{6}x^2) \\
&= \frac{1}{4}xx + \frac{1}{6}xx^2 + \frac{1}{12}xx^2 + \frac{1}{6}x^2x + \frac{1}{9}x^2x^2 + \\
&\quad \frac{1}{18}x^2x^2 + \frac{1}{12}x^2x + \frac{1}{18}x^2x^2 + \frac{1}{36}x^2x^2 \\
&= \frac{1}{4}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4,
\end{aligned}
$$

which corresponds to the distribution of heights in towers made of two independently chosen blocks.

Another probabilistic example

Consider again red, green and blue blocks, as before. We build a random tower by starting with the empty tower, and performing repeatedly and independently the step described below, until we stop.
In each step, we choose exactly one of the following actions, according to the given probabilities

- Stop building, with probability $p_s > 0$.

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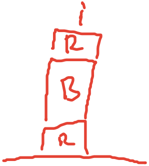
We assume $p_s + p_r + p_g + p_b = 1$.



Another probabilistic example

Consider again red, green and blue blocks, as before. We build a random tower by starting with the empty tower, and performing repeatedly and independently the step described below, until we stop.
In each step, we choose exactly one of the following actions, according to the given probabilities

- Stop building, with probability $p_s > 0$.
- Add a red block to the top, with probability $p_r \geq 0$.
- Add a green block to the top, with probability $p_g \geq 0$.
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We assume $p_s + p_r + p_g + p_b = 1$.

For each tower $\alpha \in \mathcal{T}$, let $w(\alpha)$ be the probability that we build precisely this tower and stop. Let $T_w(x) = \mathrm{OGF}(\mathcal{T}, w)$. Goal: formula for $T_w(x)$.

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Solution:

$$T_w(x) = \overset{\emptyset}{p_s} + \overset{\boxed{R}}{p_r x T_w(x)} + \overset{\boxed{G}}{p_g x^2 T_w(x)} + \overset{\boxed{B}}{p_b x^2 T_w(x)},$$

and hence $T_w(x) = \frac{p_s}{1 - p_r x - (p_g + p_b)x^2}$. "$T_w(1) = 1$"

Parametrized example

Question: What is the total number of red ~~pieces~~ blocks in all the towers of height n?

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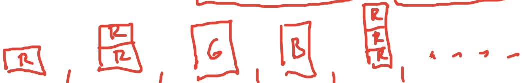
$$T(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} t_{n,k} x^n y^k = \sum_{\alpha \in \mathcal{T}} x^{|\alpha|} y^{r(\alpha)}. \in (\mathbb{Z}[[y]])[[x]]$$

$\emptyset$,



$$T(x,y) = 1 + xy + x^2y^2 + x^2 + x^2 + x^3y^3 + \ldots$$

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- $B(x,y) := \mathrm{OGF}(\mathcal{B}, w) = xy + 2x^2$ $= xy + x^2 + x^2$

$\boxed{R}$ $\boxed{G}$ $\boxed{B}$

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$\boxed{R}$ $\{\text{red}\} \times \mathcal{T}$

$xy \cdot T(x,y)$

Let us find a formula for $T(x,y)$:

$\emptyset$

$$T(x,y) = 1 + xyT(x,y) + x^2T(x,y) + x^2T(x,y) = 1 + P(x,y)T(x,y).$$

Hence $T(x,y) = \frac{1}{1-P(x,y)} = \frac{1}{1-(xy+2x^2)}$.

$\boxed{G}$ $\boxed{B}$

$T(x,1) = \text{OGF}(\mathcal{T}) = \frac{1}{1-x-2x^2}$

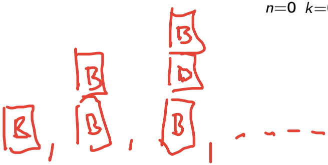
Parametrized example (continued)

Question (reminder): What is the total number of red pieces in all the towers of height n?

We have seen that $T(x,y) = \frac{1}{1-xy-2x^2}$ where

$$T(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t_{n,k} x^n y^k = \sum_{\alpha\in\mathcal{T}} x^{|\alpha|} y^{r(\alpha)}.$$

$x=1$: $\dfrac{1}{1-y-2} = \dfrac{1}{-1-y}$



x^2, x^4, x^6,

1 1 1

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Define $D(x,y) := \frac{\mathrm{d}}{\mathrm{d}y} T(x,y) = \frac{x}{(1-xy-2x^2)^2}$.

Observe:

$$D(x,y) = \sum_{n=0}^{\infty}\sum_{\underset{k=1}{\underline{k=0}}}^{\infty} k t_{n,k} x^n y^{\underline{k-1}} = \sum_{\alpha\in\mathcal{T}} r(\alpha) x^{|\alpha|} y^{r(\alpha)-1}.$$

$$[x^n] D(x,y) = 1\cdot t_{n,1} y^0 + 2\cdot t_{n,2} y^1 + 3\cdot t_{n,3} y^2 + \dots + n\, t_{n,n} y^{n-1} \xrightarrow{y=1} \underline{t_{n,1} + 2\cdot t_{n,2} + 3 t_{n,3} + \dots + n\, t_{n,n}}$$

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Hence:

$$D(x,1) = \frac{x}{(1-x-2x^2)^2} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} k t_{n,k}x^n = \sum_{\alpha\in\mathcal{T}} r(\alpha) x^{|\alpha|}.$$

Answer to the question: $[x^n]D(x,1) = [x^n]\frac{x}{(1-x-2x^2)^2}$.