1. Let $d > 1$ and let $C_d(h), h \in \mathbb{R}^d$, be an autocovariance function of a random field $\{Z(u), u \in \mathbb{R}^d\}$, i.e. C_d is a positive semidefinite function on $\mathbb{R}^d$. Assume that the random field is stationary and isotropic and hence $C_d(h) = f(\|h\|_d), h \in \mathbb{R}^d$, for some function $f : [0, \infty) \to \mathbb{R}$, where $\|h\|_d$ denotes the d-dimensional Euclidean norm. For $1 \le k < d$ define $C_k(u) = f(\|u\|_k), u \in \mathbb{R}^k$. Prove that C_k is the autocovariance function of some random field $\{Y(u), u \in \mathbb{R}^k\}$.

Stationarity: $C_d(x,y) = C_d(\overset{(h)}{\underbrace{y-x}_{\in\mathbb{R}^d}})$, $x,y\in\mathbb{R}^d$

Isotropy: $C_d(x,y) = C_d(\|y-x\|_d)$

assume: $C_d(h) = f(\underbrace{\|h\|_d}_{\in \mathbb{R}_0^+})$, $h\in\mathbb{R}^d$ $\to$ is PSD, $f : [0,\infty) \to \mathbb{R}$

define $C_k(u) = f(\|u\|_k)$, $u\in\mathbb{R}^k$

$1 \le k < d$ ↳ is this positive semidefinite?

$u \in \mathbb{R}^k = (u_1,\dots,u_k)$ $\to$ $h\in\mathbb{R}^d = (u_1,u_2,\dots,u_k,\underbrace{0,\dots,0}_{d-k})$

we want to check: $\forall n\in\mathbb{N}$ $\forall v_1,\dots,v_n \in\mathbb{R}^k$ $\forall \alpha_1,\dots,\alpha_n \in (\mathbb{C})\,\mathbb{R}$

$$\sum_{i=1}^n\sum_{j=1}^n \alpha_i\bar{\alpha}_j f(\underbrace{\|v_i-v_j\|_k}_{\|h_i-h_j\|_d}) = \sum\sum \alpha_i\alpha_j f(\|h_i-h_j\|_d) \ge 0$$

↑ (assu… $f(\|\cdot\|_d$ …

does not work for $k>d$! (see Ex. 3)

works also with $_2\|h\|_d = (h_1^2+\dots+h_d^2)^{1/2}$

$_1\|h\|_d = |h_1|+\dots+|h_d|$

$_\infty\|h\|_d = \max\{|h_1|,\dots,|h_d|\}$

and others, provided they have a direct analogue in $\mathbb{R}^k$, $k<d$.