7. Let Z be a Markov random field on a lattice L and with respect to the relation $i \sim j$. Assume that the random variables $\{Z_i, i \in L\}$ are binary, i.e. $S = \{0, 1\}$, and that Z_i have the same expectation. We want to test the null hypothesis of independence of $\{Z_i, i \in L\}$, taking into account the neighbourhood relation $i \sim j$. Under the assumptions above, the null hypothesis in fact states that $\{Z_i, i \in L\}$ are i.i.d. random variables.

a) Propose an appropriate test statistics T;

b) discuss how to perform the test if we can simulate from the model under the null hypothesis;

c) discuss how to perform the test if we cannot simulate from the model under the null hypothesis;

d) if the point $i \in L$ has many neighbours and the point $j \in L$ has few neighbours, the impact of Z_i on the value of T can perhaps be much higher than the impact of Z_j – propose a way how to compensate for that.

a) $BB = \frac{1}{2} \sum_{i \in L} \sum_{j \in L} w_{ij} Z_i Z_j$, $BW = \frac{1}{2} \sum_{i \in L} \sum_{j \in L} w_{ij} (Z_i - Z_j)^2$

(WW) $(1-Z_i)(1-Z_j)$

$w_{ij} = 0 \Leftarrow i \not\sim j$

b) „analytical test": reject iff $T_0 \notin [w_{\frac{\alpha}{2}}, w_{1-\frac{\alpha}{2}}]$ known d

„Monte Carlo test": $w_{\frac{\alpha}{2}}, w_{1-\frac{\alpha}{2}}$ estimated from simulations under H_0

$Z = (Z_1, \dots, Z_n)$ ... observed data $\to T_0$

simulate $Z^1, \dots, Z^M$ under H_0 $\to T_1, \dots, T_M$, $M \in \mathbb{N}$

reject iff T_0 is extreme among $(T_0, T_1, \dots, T_M)$

(very general, not specific to H_0 of independence)

c) spec. for H_0: Z_i i.i.d. ... use bootstrap („sampling from empirical dist. of th

simulation $\equiv$ sampling from true F

bootstrap $\equiv$ $\hat{F}$

↳ with replacement .. binomial sampling

↳ without replacement (permutation)

--- hypergeometric sampl

d) we can use normalized weights : $w_{i\sim j} = \frac{1}{|\partial i|} \cdot \mathbb{1}(i \sim j$

(not symmetric weig

instead of binary weights : $w_{i\sim j} = \mathbb{1}(i \sim j)$

(symmetric)