NMFM 402 , Practical 2 , GLM – EDF, MLE

Ex. 1 a) $f(y|\lambda) = \frac{\lambda^y e^{-\lambda}}{y!} = \exp\left\{\log\left(\frac{\lambda^y}{y!}\right) - \lambda\right\} =$

$= \exp\left\{\frac{y \cdot \log\lambda - \lambda}{1} - \log(y!)\right\} = \exp\left\{\frac{y\cdot\theta - e^\theta}{\varphi} - \log(y!)\right\}$

where: $\theta = \log\lambda$, $\varphi = 1$, $w = 1$

$b(\theta) = e^\theta$ ... cumulant function

$c(y, \varphi, w) = -\log(y!)$

verify: • $\lambda \in (0, +\infty) \Rightarrow \theta = \log\lambda \in \mathbb{R} = \Theta$ ... an open set ✓

• $b(\theta) = e^\theta$ ... twice differentiable ✓

• $b'(\theta) = e^\theta$ ... invertible ✓

$\Longrightarrow$ Poisson distr. belongs to EDF.

b) using (2.3.) in [1] : $\underline{EY} = b'(\theta) = \underline{e^\theta = \lambda}$
and (2.4.) in [1] : $\underline{\operatorname{var} Y} = \frac{\varphi}{w} \cdot b''(\theta) = \underline{e^\theta = \lambda}$

c) either by (2.5) : $\operatorname{var} Y = \frac{\varphi}{w} v(\mu) = v(\mu)$,
and $\mu = EY = e^\theta = \lambda$
and from (b) $\operatorname{var} Y = e^\theta = \lambda$
$\Longrightarrow \underline{v(\mu) = \mu}$

or, by definition:

$\underline{v(\mu)} = b''\left((b')^{-1}(\mu)\right) = \exp\{\log(\mu)\} = \underline{\mu}$

d) canonical link; see (2.8): $g(\mu) = (b')^{-1}(\mu) = \log \mu$

Ex. 2 (a) by definition

$$f(y_1 \dots y_n | \theta_1, \dots, \theta_n) = \prod_{i=1}^{n} \exp\{ y_i \theta_i - e^{\theta_i} - \log(y_i!) \},$$

so $\underline{\underline{l(\theta_1 \dots \theta_n | y_1 \dots y_n) = \sum_{i=1}^{n} (y_i \theta_i - e^{\theta_i}) - \sum_{i=1}^{n} \log(y_i!)}}$

(compare with (2.10))

(b) logarithmic link function:

$$\sum_{j=0}^{k} x_{ij} \beta_j = \log \mu_i = \log e^{\theta_i} = \theta_i$$

in result: $l(\beta_0, \dots, \beta_k | y_1, \dots, y_n) = \sum_{i=1}^{n} \left[y_i \left(\sum_{j=0}^{k} x_{ij} \beta_j \right) - e^{\sum_{j=0}^{k} x_{ij} \beta_j} \right] - \sum_{i=1}^{n} \log(y_i!)$

(c) 1. MLE: $\frac{\partial l}{\partial \beta_j} = 0, \; j = 0 \dots k$

$$\sum_{i=1}^{n} \left[y_i \cdot x_{ij} - e^{\sum_{j=0}^{k} x_{ij} \beta_j} \cdot x_{ij} \right] = 0, \quad j = 0, \dots, k$$

2. by (2.14) with $\mu_i = e^{\sum_{j=0}^{k} x_{ij} \beta_j}$, $w_i = 1$,

$v(\mu_i) = \mu_i$ and $g'(\mu_i) = \frac{1}{\mu_i}$:

$$\sum_{i=1}^{n} \frac{y_i - e^{\sum_{j=0}^{k} x_{ij} \beta_j}}{1} \cdot x_{ij} = 0, \quad j = 0, \dots, k$$

Ex. 3. a) $f(y \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} =$

$$= \exp\left\{ \frac{-\frac{\beta}{\alpha} y + \log \frac{\beta}{\alpha}}{\frac{1}{\alpha}} + \frac{\log \alpha}{\frac{1}{\alpha}} - \log \Gamma(\alpha) + (\alpha-1)\log y \right\}$$

$$= \exp\left\{ \frac{\theta y - b(\theta)}{\varphi} + c(y, \varphi) \right\}, \text{ with}$$

$\theta = -\frac{\beta}{\alpha}$, $\varphi = \frac{1}{\alpha}$, $w = 1$

$b(\theta) = -\log(-\theta)$ ... cumulant function

$c(y, \varphi) = -\frac{\log \varphi}{\varphi} - \log \Gamma\left(\frac{1}{\varphi}\right) + \left(\frac{1}{\varphi} - 1\right) \log y$

verify: • $\alpha > 0, \beta > 0 \Rightarrow \theta = -\frac{\beta}{\alpha} < 0 \Rightarrow \Theta = (-\infty, 0)$ ... an open set ✓

• $b(\theta) = -\log(-\theta)$ ... 2x differentiable

• $b'(\theta) = -\frac{1}{\theta}$ ... invertible

$\Rightarrow$ Gamma distr. belongs to EDF.

b) $EY = b'(\theta) = -\frac{1}{\theta} = \frac{\alpha}{\beta} = \mu$

$\operatorname{var} Y = \frac{\varphi}{w} b''(\theta) = \varphi \cdot \frac{1}{\theta^2} = \frac{\alpha}{\beta^2}$

c) $v(\mu) = b''\left[(b')^{-1}(\mu)\right] = \frac{1}{\left(-\frac{1}{\mu}\right)^2} = \mu^2$

d) canonical link: $g(x) = (b')^{-1}(x) = -\frac{1}{x}$

Ex. 4 a) $\cancel{l = \frac{1}{\varphi}\sum_{i=1}^{n}(y}$

$$l(\theta_1,\dots,\theta_n \mid y_1,\dots,y_n) = \frac{1}{\varphi}\sum_{i=1}^{n}\left[y_i\theta_i + \log(-\theta_i)\right] + \sum_{i=1}^{n} c(y_i,\varphi)$$

b) link function $g(x) = \log(x)$ ... not canonical

$$\theta_i = \log(\mu_i) = \log\left(-\frac{1}{\theta_i}\right) = e^{\sum_{j=0}^{k} x_{ij}\beta_j}$$

$$\theta_i = -e^{-\sum_{j=0}^{k} x_{ij}\beta_j}$$

$$l(\beta_0,\dots,\beta_k \mid y_1,\dots,y_n) = \frac{1}{\varphi}\sum_{i=1}^{n}\left[-y_i e^{-\sum_{j=0}^{k} x_{ij}\beta_j} - \sum_{j=0}^{k} x_{ij}\beta_j\right] + \sum_{i=1}^{n} c(y_i,\varphi)$$

c) $$\frac{1}{\varphi}\sum_{i=1}^{n}\left[y_i e^{-\sum_{j=0}^{k} x_{ij}\beta_j} - 1\right]x_{il} = 0, \quad l = 0,\dots,k$$

Ex. 5 a) for normal distr. $v(\mu) = 1 = \mu^0$
$\Rightarrow$ belongs to Tweedie model with $p = 0$

b) for Poisson distr. $v(\mu) = \mu$
$\Rightarrow$ belongs to Tweedie model with $p = 1$

c) for gamma distr. $v(\mu) = \mu^2$
$\Rightarrow$ belongs to Tweedie model with $p = 2$