Analytic combinatorics

Lecture 1

March 10, 2021

About the course

- Me: Vít Jelínek

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- Contact: jelinek@iuuk.mff.cuni.cz

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Notation:

- $\mathbb{N}$: natural numbers, i.e., $\{1, 2, 3, \dots\}$
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q}$: rational numbers
- $\mathbb{R}$: real numbers
- $\mathbb{C}$: complex numbers

Overview of analytic method(s) in combinatorics

Basic situation: Suppose we have a set $\mathcal{S}$ of some combinatorial objects (graphs, permutations, set partitions, ...) for which we have a notion of size. We want to determine or estimate the number s_n of objects of size n in $\mathcal{S}$. But finding a formula for s_n directly is impossible.

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The "analytic" approach:

1. Find a formula for the generating function of $\mathcal{S}$, which is a formal power series
$$S(x) = \sum_{n=0}^{\infty} s_n x^n \text{ or maybe } S(x) = \sum_{n=0}^{\infty} s_n \frac{x^n}{n!}.$$

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2. Treat $S(x)$ as an actual function from $\mathbb{C}$ to $\mathbb{C}$.

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2. Treat $S(x)$ as an actual function from $\mathbb{C}$ to $\mathbb{C}$.
3. Apply complex-analytic tools (analytic continuation, contour integrals, residues, ...) to the function $S(x)$ to estimate s_n.

Formal power series

For the rest of today's lecture, fix a coefficient ring K, to be a commutative ring with a multiplicative unit and with no zero divisors. (Imagine $K = \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{C}$.)

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A sequence $(a_0, a_1, a_2, \dots) = (a_n)_{n=0}^{\infty}$ of elements of K, can be represented by a formal power series (in x)

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n.$$

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Then a_n is the coefficient of degree n in the f.p.s. $A(x)$, denoted by $[x^n]A(x)$.

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Let $K[[x]]$ denote the set of all f.p.s. in x over K.

Operations with f.p.s.

Consider $A(x), B(x) \in K[[x]]$, with $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$.
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\begin{aligned}
A(x) + B(x) &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots \\
A(x)B(x) &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.
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Observe:

- The series $0 = 0 + 0x + 0x^2 + \cdots$ satisfies $A(x) + 0 = 0 + A(x) = A(x)$.

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- The series $1 = 1 + 0x + 0x^2 + \cdots$ satisfies $A(x) \cdot 1 = 1 \cdot A(x) = A(x)$.
- In fact, $K[[x]]$ is a commutative ring with a unit (and no zero divisors).

Multiplicative inverses

Definition

Let $A(x)$ be a f.p.s. from $\in K[[x]]$. A multiplicative inverse (or reciprocal) of $A(x)$ is a f.p.s. $B(x) \in K[[x]]$ such that $A(x)B(x) = 1$. The multiplicative inverse of $A(x)$ (if it exists) is denoted $A(x)^{-1}$ or $\frac{1}{A(x)}$.

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When does a f.p.s. have a multiplicative inverse?

Existence of inverses

Lemma

A f.p.s. $A(x) = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$ *has a multiplicative inverse in* $K[[x]]$ *if and only if the coefficient* $a_0 = [x^0]A(x)$ *has a multiplicative inverse in* K*. The inverse, when it exists, is unique.*

Formal convergence

Let $A_0(x), A_1(x), A_2(x), \ldots$ be an infinite sequence of f.p.s. from $K[[x]]$. How to define its limit $\lim_{k\to\infty} A_k(x)$? (Problem: we cannot assume that there is any notion of convergence for elements of K.)

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Definition

A f.p.s. $L(x) \in K[[x]]$ is the (formal) limit of the sequence $A_0(x), A_1(x), A_2(x), \ldots$, if for every $n \in \mathbb{N}_0$ there is a $k_0 \in \mathbb{N}_0$ such that for all $k \geq k_0$ we have

$$[x^n]A_k(x) = [x^n]L(x).$$

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Examples:

- The sequence of f.p.s. $1 + x, 1 + x^2, 1 + x^3, \ldots$ has limit 1.

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Examples:

- The sequence of f.p.s. $1+x, 1+x^2, 1+x^3, \ldots$ has limit 1.
- The sequence of f.p.s. $1+x, 1+\frac{x}{2}, 1+\frac{x}{3}, \ldots$ does not converge to a limit.

Summing infinitely many f.p.s.

Let $A_0(x), A_1(x), A_2(x), \ldots$ be an infinite sequence of f.p.s. from $K[[x]]$. How to define their infinite sum

$$A_0(x) + A_1(x) + A_2(x) + \cdots ?$$

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Answer: as a limit of the sequence partial sums
$A_0(x), A_0(x) + A_1(x), A_0(x) + A_1(x) + A_2(x), \dots$.

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Answer: as a limit of the sequence partial sums $A_0(x), A_0(x) + A_1(x), A_0(x) + A_1(x) + A_2(x), \dots$.

Observe: $A_0(x) + A_1(x) + A_2(x) + \cdots$ exists iff for every degree $n \in \mathbb{N}_0$, there are only finitely many summands $A_k(x)$ with $[x^n]A_k(x) \neq 0$.

Examples of infinite sums

Example 1: Consider

$$
\begin{aligned}
A_0(x) &= 1 + x + x^2 + x^3 + x^4 + \cdots \\
A_1(x) &= x + x^2 + x^3 + x^4 + \cdots \\
A_2(x) &= x^2 + x^3 + x^4 + \cdots \\
A_3(x) &= x^3 + x^4 + \cdots \\
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Is the sum $A_0(x) + A_1(x) + A_2(x) + \cdots$ defined? What is its value?

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Is the sum $A_0(x) + A_1(x) + A_2(x) + \cdots$ defined? What is its value?

Example 2: For which $B(x) \in K[[x]]$ is the sum $B(x) + B^2(x) + B^3(x) + B^4(x) + \cdots$ defined?

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Is the sum $A_0(x) + A_1(x) + A_2(x) + \cdots$ defined? What is its value?

Example 2: For which $B(x) \in K[[x]]$ is the sum $B(x) + B^2(x) + B^3(x) + B^4(x) + \cdots$ defined? Answer: Sum is defined iff $[x^0]B(x) = 0$.

Composition

Definition

For two f.p.s. $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$, their composition, denoted $(A \circ B)(x)$ or $A(B(x))$, is the f.p.s. defined as the infinite sum

$$\sum_{n=0}^{\infty} a_n B^n(x) = a_0 + a_1 B(x) + a_2 B^2(x) + a_3 B^3(x) + \cdots .$$

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When is $A(B(x))$ defined?

Composition

Lemma

$A(B(x))$ exists iff at least one of these two conditions holds:

1. *$A(x)$ is a polynomial (i.e., has only finitely many nonzero coefficients).*
2. $[x^0]B(x) = 0$.

Composition

Lemma

$A(B(x))$ exists iff at least one of these two conditions holds:

1. *$A(x)$ is a polynomial (i.e., has only finitely many nonzero coefficients).*
2. *$[x^0]B(x) = 0$.*

Definition

A f.p.s. $B(x)$ is composable if $[x^0]B(x) = 0$.

Nasty examples

Nasty example 1. Composition is not continuous, i.e., $\lim_{k\to\infty} A_k(x) = L(x)$ does NOT imply $\lim_{k\to\infty} A_k(B(x)) = L(B(x))$, even when all the expressions are defined:

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Nasty example 2. Composition is not associative: take $A(x) = 1 - \sum_{n=1}^{\infty} \binom{2n-2}{n-1}/(n2^{2n-1})x^n$ (Taylor series of $\sqrt{1-x}$), $B(x) = 2x - x^2$, $C(x) = 2$.

Composing composable series is not nasty

The good news: No such nastyness can occur for a composition $A(B(x))$ with B composable.

Lemma

- *If* $A_0(x), A_1(x), A_2(x), \ldots$ *is a sequence of f.p.s. with limit* $A(x)$*, and if* $B_0(x), B_1(x), B_2(x), \ldots$ *is a sequence of composable f.p.s. with limit* $B(x)$ *(which is necessarily also composable), then* $\lim_{k\to\infty} A_k(B_k(x)) = A(B(x))$.
- *If* $A(x)$*,* $B(x)$ *and* $C(x)$ *are f.p.s., with* $B(x)$ *and* $C(x)$ *composable, then* $(A \circ B) \circ C = A \circ (B \circ C)$.

Composition inverse

Observe: The series $\mathrm{Id}(x) = x$ is the neutral element for composition: $\mathrm{Id}(A(x)) = A(\mathrm{Id}(x)) = A(x)$.

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Definition

Let $A(x) \in K[[x]]$ be composable. A *(left) composition inverse* of $A(x)$ is a f.p.s. $B(x)$ such that $B(A(x)) = x$. It is denoted $A^{\langle -1 \rangle}(x)$.

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Lemma

A composable f.p.s. $A(x) \in K[[x]]$ has a composition inverse if and only if the coefficient $[x^1]A(x)$ has a (multiplicative) inverse in K. In such case, the composition inverse $B(x) = A^{\langle -1 \rangle}(x)$ is unique, is composable, and satisfies $B^{\langle -1 \rangle}(x) = A(x)$.