2. Show that a Markov chain $\{Z_1,\dots,Z_n\}$ is a Markov random field with respect to the relation $i \sim j \Leftrightarrow |i-j| \le 1$. Prove that the converse implication holds as follows: if $\{Z_1,\dots,Z_n\}$ is a Markov random field with a probability density function satisfying $p(\boldsymbol{z})>0$ for all $\boldsymbol{z}=(z_1,\dots,z_n)^T$ then it is a Markov chain.

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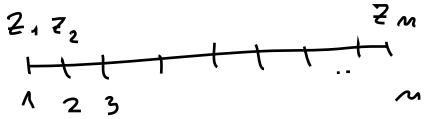

M. Chain: $p(z_i \mid z_{i-1},\dots,z_1) = p(z_i \mid z_{i-1})$

M.R.F.: $p(z_i \mid z_n,\dots,z_{i+1},z_{i-1},\dots,z_1) =$

$= p(z_i \mid z_{i+1}, z_{i-1})$

$L=\{1,\dots,n\}$, $i \sim j \Leftrightarrow |i-j|=1$

a) M.C. $\overset{?}{\Rightarrow}$ M.R.F.

$1<i<n$: $\quad p(z_i \mid z_{-i}) = \dfrac{p(z_1,\dots,z_n)}{p(z_{-i})} = (*)$

$p(z_{-i})>0$

↳ $p(z_1,\dots,z_n) = p(z_n \mid z_{n-1},\dots,z_1)\cdot p(z_{n-1} \mid z_{n-2},\dots,z_1)\cdot\ldots\cdot p(z_2 \mid z_1)\cdot p(z_1)$

$\overset{M.C.}{=} p(z_n \mid z_{n-1})\cdot\ldots\cdot p(z_2 \mid z_1)\cdot p(z_1)$

↳ $p(z_{-i}) = \int_S p(z_1,\dots,z_n)\,\nu_i(dz_i)$ $\qquad [\nu_i=\nu]$

$$(*) = \frac{p(z_n \mid z_{n-1})\cdot\ldots\cdot p(z_2 \mid z_1)\cdot p(z_1)}{\left(\int_S p(z_{i+1} \mid w_i)\,p(w_i \mid z_{i-1})\,\nu_i(dw_i)\right) p(z_n \mid z_{n-1})\ldots p(z_{i+2} \mid z_{i+1})\cdot p(z_{i-1} \mid z_{i-2})\cdot\ldots\cdot p(z_1)}$$

$$= \frac{p(z_{i+1} \mid z_i)\,p(z_i \mid z_{i-1})}{\int_S p(z_{i+1} \mid w_i)\,p(w_i \mid z_{i-1})\,\nu_i(dw_i)} = (**)$$

$= p(z_{i+1} \mid z_i)\,p(z_i \mid z_{i-1})\cdot p(z_{i-1})$

$$p(z_i \mid z_{i+1}, z_{i-1}) = \frac{p(z_{i+1}, z_i, z_{i-1})}{\int_S p(z_{i+1}, w_i, z_{i-1})\,\nu_i(dw_i)} \overset{M.C.}{=}$$

$$= \frac{p(z_{i+1} \mid z_i)\,p(z_i \mid z_{i-1})\,\cancel{p(z_{i-1})}}{\cancel{p(z_{i-1})}\int_S p(z_{i+1} \mid w_i)\,p(w_i \mid z_{i-1})\,\nu_i(dw_i)} = (**)$$

$\Rightarrow$ M.R.F. property holds with "$\sim$"

• for $i=1$, $i=n$ similarly

b) assume $Z \sim$ M.R.F. on L with $i \sim j \Leftrightarrow |i-j| = 1$

$p(z) > 0 \quad \forall z \in S^L$.

We need to factorize joint densities: Hammersley, Clifford theorem

$$p(z) = \prod_{c \in \mathcal{C}} g_c(z_c) \overset{\text{for our "}\sim\text{"}}{=} g_\phi() \cdot \left(\prod_{i=1}^{n} g_i(z_i)\right)\left(\prod_{i=1}^{n-1} g_{i,i+1}(z_i, z_{i+1})\right) = (*$$

$\hookrightarrow z_c = (z_i,\ i \in c)$

then: $\underbrace{p(z_l \mid z_{l-1}, \dots, z_1)}_{l = 2, \dots, n} = \dfrac{p(z_l, \dots, z_1)}{p(z_{l-1}, \dots, z_1)} = \quad \overset{?}{=} \underline{p(z_l \mid z_{l-1})}$

$$= \frac{\int p(z_1, \dots, z_n)\, \nu(dz_n) \dots \nu(dz_{l+1})}{\int p(z_1, \dots, z_n)\, \nu(z_n) \dots \nu(dz_{l+1})\, \nu(dz_l)} \overset{(*)}{=}$$

$$= \frac{\int \left(\prod_{j=l}^{n} g_j(z_j)\right)\left(\prod_{j=l-1}^{n} g_{j,j+1}(z_j, z_{j+1})\right) \nu(dz_n) \dots \nu(dz_{l+1})}{\int \left(\prod_{j=l}^{n} g_j(z_j)\right)\left(\prod_{j=l-1}^{n} g_{j,j+1}(z_j, z_{j+1})\right) \nu(dz_n) \dots \nu(dz_{l+1})\, \nu(dz_l)} =$$

cancelling was ok $\dots p(z) > 0 \quad \forall z \in S^L$ $\qquad = \underline{f(z_l, z_{l-1})}$

$$p(z_l, \dots, z_1) = p(z_l \mid z_{l-1}, \dots, z_1) \cdot p(z_{l-1}, \dots, z_1) =$$

$$= \overbrace{f(z_l, z_{l-1})} \cdot p(z_{l-1}, \dots, z_1)$$

$$p(z_l \mid z_{l-1}) = \frac{p(z_l, z_{l-1})}{p(z_{l-1})} = \frac{\int f(z_l, z_{l-1}) \cdot p(z_{l-1}, \dots, z_1)\, \nu(z_{l-2}) \dots \nu(}{p(z_{l-1})}$$

$$= \frac{f(z_l, z_{l-1}) \cdot p(z_{l-1})}{p(z_{l-1})} = f(z_l, z_{l-1}) \ .$$