5. Consider a Gaussian random field on a lattice L, i.e. the joint distribution of $\{Z_i, i \in L\}$ is n-dimensional Gaussian. Assume the covariance matrix $\boldsymbol{\Sigma}$ is regular and hence $\boldsymbol{Q} = \boldsymbol{\Sigma}^{-1}$ exists. The joint probability density function of $\{Z_i, i \in L\}$ is

$$-\tfrac{1}{2}(z-\mu)Q(z-\mu)$$

$$p(\boldsymbol{z}) = \frac{\sqrt{\det \boldsymbol{Q}}}{(2\pi)^{n/2}} \exp\left\{ -\frac{1}{2} \sum_{i,j \in L} q_{ij}(z_i - \mu_i)(z_j - \mu_j) \right\}, \quad \boldsymbol{z} \in \mathbb{R}^L.$$

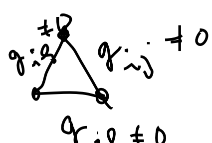
If this random field is to be Markov, what should be the neighbourhood relation?

$$p(z) = \underbrace{\frac{\sqrt{\det Q}}{(2\pi)^{n/2}}}_{g_\phi()} \cdot \underbrace{\exp\Big\{-\frac{1}{2}\sum_{i \in L} q_{ii}(z_i - \mu_i)^2\Big\}}_{\prod_{i=1}^{n} g_i(z_i)} \cdot \underbrace{\exp\Big\{-\frac{1}{2}\sum_{\substack{i,j \in L \\ i \neq j}} q_{ij}(z_i - \mu_i)\ldots}_{\prod_{\substack{i,j \in L \\ i \neq j}} g_{ij}(z_i, z_j)}$$

$p(z) > 0 \quad \forall z \in S^L$

↳ clique ↳ cliques ↳ ?

$Q = (q_{ij})$ determines "$\sim$" ... $\boxed{i \sim j \iff q_{ij} \neq 0}$ (NR)

? 

$g_{ij\ell}(z_i, z_j, z_\ell) = 1$

(NR) may generate cliques with more than 2 points, for those we put $g_c(z_c) = 1$ and include them in

+ H.-C. theorem $\Rightarrow$ Markov R.F.

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directly: $p(z_i \mid z_{-i}) = \ldots$

$$Z_i \mid Z_{-i} \sim N\Big(\mu_i - \frac{1}{q_{ii}} \sum_{j:\, j \neq i} q_{ij}(z_j - \mu_j),\ \frac{1}{q_{ii}}\Big)$$

Markov if ... $\partial i = \{j : q_{ij} \neq 0\}$ $=$ (NR)

$i_1 \sim i_2$ even if $q_{i_1 i_2} = 0$ $\qquad Z_i \mid Z_{-i}$ "=" $Z_i \mid Z_{\partial i}$