6. Local characteristics do not determine the joint distribution. Consider a lattice with two lattice points $L = \{i, j\}$ and assume that $Z_i \mid Z_j = z_j$ has an exponential distribution with rate z_j and $Z_j \mid Z_i = z_i$ has an exponential distribution with rate z_i. Show that these conditional distributions do not correspond to any probability distribution, i.e. a (proper) joint probability density function of the vector $(Z_i, Z_j)^T$ does not exist.

$i \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ\, j$

$$Z_i \mid Z_j = z_j \sim \mathcal{E}xp(z_j)$$

$$Z_j \mid Z_i = z_i \sim \mathcal{E}xp(z_i)$$

(Z_i, Z_j) does not exist such that holds

$$p(z_j \mid z_i) = z_i\, e^{-z_i z_j}, \quad z_j > 0 \qquad \Big| \quad p_j(z_j)$$

$$p(z_i \mid z_j) = z_j \cdot e^{-z_i z_j}, \quad z_i > 0 \qquad \Big| \quad p_i(z_i)$$

if joint density exists: $p(z_i, z_j) = p(z_j \mid z_i) \cdot p_i(z_i)$

$$= p(z_i \mid z_j) \cdot p_j(z_j)$$

$$z_i\, \cancel{e^{-z_i z_j}}\, p_i(z_i) = z_j\, \cancel{e^{-z_i z_j}}\, p_j(z_j)$$

$$z_i\, p_i(z_i) = z_j\, p_j(z_j) \qquad \forall\, z_i, z_j > 0$$

fix $z_i = x$ ... $\underbrace{x \cdot p_i(x)}_{c} = z_j\, p_j(z_j) \Rightarrow p_j(z_j) = \frac{c}{z_j}$,

$\Rightarrow p_j$ not integrable,

not a p.d.f.

$\Rightarrow p_i$ the same