Quasigroups, 3-nets and isotopy

Definition of a quasigroup. Let $\cdot$ be a binary operation upon a set Q. For every $a \in Q$ define $L_a : Q \to Q$ and $R_a : Q \to Q$ by

$$L_a : x \mapsto ax \text{ and } R_a : x \mapsto xa.$$

Call L_a the *left translation* of the element a, and R_a the *right translation.*

The pair $(Q, \cdot)$ is called a *quasigroup* if L_a and R_a permute Q for each $a \in Q$. There are many alternative definitions of a quasigroup. We shall get to them later.

Operations of Q will be denoted by different symbols. For example $+$ or $*$ or $\circ$. The choice of $\cdot$ is implicit. Hence stating that Q is a quasigroup means that we are considering the pair $(Q, \cdot)$.

The application of $\cdot$ may be replaced by a juxtaposition. Thus xy is the same as $x \cdot y$. It is usual to assume that the juxtaposition binds more tightly than the explicit use of an operation. E.g., $xu \cdot (yz \cdot w)$ is the same as $(x \cdot u) \cdot ((y \cdot z) \cdot w)$.

Multiplication table. Every binary operation may be represented by its *multiplication (or operational)* table. Both

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

$*$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

are multiplication tables of a quasigroup. The operation of the quasigroup upon the left is equal to $(x + y) \bmod 3$. The formula for the operation of the quasigroup upon the right is $x * y \equiv -x - y \bmod 3$. The latter quasigroup is *idempotent*, i.e., $x * x = x$ for every $x \in Q$.

Consider the quasigroup $(\mathbb{Z}_3, +)$ and decompose it to the *border of the table* (upon the left) and the *body of the table* (upon the right):

$+$	0	1	2
0			
1			
2			

	0	1	2
	1	2	0
	2	0	1

Latin squares and quasigroups. Let S be a finite set, $|S| = n$. A *latin square* over S is an $n \times n$ matrix $A = (a_{ij})$ such that for every $i \in \{1, \dots, n\}$

$$S = \{a_{i1}, \dots, a_{in}\} = \{a_{1i}, \dots, a_{ni}\}.$$

If $\cdot$ is a binary operation upon set Q, then $(Q, \cdot)$ is a quasigroup if and only if the body of the operation table is a latin square.

Lines induced by a quasigroup. Let $(Q, \cdot)$ be a quasigroup. Put $\mathcal{P} = Q \times Q$ and treat the set $\mathcal{P}$ as a *set of points.* Define $\mathcal{L}_i$, $1 \le i \le 3$, as sets of parallel lines *(pencils)* such that $\mathcal{L}_1 = \{r_a;\ a \in Q\}$, $\mathcal{L}_2 = \{c_a;\ a \in Q\}$ and $\mathcal{L}_3 = \{s_a;\ a \in Q\}$, where

$$r_a = \{(a, x);\ x \in Q\} \quad \text{(the row of } a\text{)}$$
$$c_a = \{(x, a);\ x \in Q\} \quad \text{(the column of } a\text{)}$$
$$s_a = \{(x, y) \in Q \times Q;\ xy = a\} \quad \text{(the transversal of } a\text{)}$$

Axioms of the 3-net. The system $(\mathcal{P};\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$ clearly fulfils the following axioms:

- $\forall p\in\mathcal{P}, \forall i\in\{1,2,3\}\,\exists!\,\ell\in\mathcal{L}_i$ such that $p\in\ell$;
- $\forall i,j\in\{1,2,3\}$, where $i\neq j$: $(\ell_i\in\mathcal{L}_i,\ \ell_j\in\mathcal{L}_j\ \Rightarrow\ |\ell_i\cap\ell_j|=1)$

This can be put in words by saying that through each point there passes exactly one line of a given pencil, and that two lines from different pencils intersect in exactly one point.

Any system that fulfils the above two axioms is called a *3-net*.

Theorem. *Let $(\mathcal{P};\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$ be a 3-net. Then $|\mathcal{L}_1|=|\mathcal{L}_2|=|\mathcal{L}_3|=|\ell|$ for any $\ell\in\bigcup\mathcal{L}_i$, $i\in\{1,2,3\}$.*

Proof. Suppose that $1\le i<j\le 3$, $\ell_i\in\mathcal{L}_i$, $\ell_j\in\mathcal{L}_j$ and $\{1,2,3\}=\{i,j,k\}$. Map ℓ_i upon ℓ_j in the following way: take $q\in\ell_i$ and consider the line $\ell_k\in\mathcal{L}_k$ that passes through q. This line intersects ℓ_j in a point, say q'. The mapping $q\mapsto q'$ is a bijection since through every point of ℓ_j there passes exactly one line of $\mathcal{L}_k$.

The mapping $q\mapsto q'$ thus also proves that $|\mathcal{L}_k|=|\ell_i|$. If ℓ_i' is another line from $\mathcal{L}_i$, then $|\mathcal{L}_k|=|\ell_i'|=|\ell_j|$ by the same argument. □

Coordinatization. Let $(\mathcal{P};\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$ be a 3-net, and let Q be a set of the same cardinality as $\mathcal{L}_i$, $1\le i\le 3$. Suppose that $\mu_i\colon Q\to\mathcal{L}_i$ are bijections. If $x,y\in Q$ then there exists a unique line in $\mathcal{L}_3$ that passes through the intersection of $\mu_1(x)$ and $\mu_2(y)$. This line is equal to some $\mu_3(z)$. Hence there exists a binary operation upon Q such that

$$xy=z\ \Leftrightarrow\ \mu_1(x)\cap\mu_2(y)\cap\mu_3(z)\neq\emptyset.. \tag{C}$$

The operation is a quasigroup since knowledge of y and z determines x uniquely, and, similarly, knowledge of x and z determines y uniquely.

Let Q be a quasigroup and let $\mu_i\colon Q\to\mathcal{L}_i$ be a bijection for each $i\in\{1,2,3\}$. If (C) holds for all $x,y,z\in Q$, then (μ_1,μ_2,μ_3) is called a *coordinatization* of the 3-net $(\mathcal{P};\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$.

Proposition. *Let $(\mathcal{P};\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$ be a 3-net, and let Q and Q' be quasigroups. If $\mu_i\colon Q\to\mathcal{L}_i$ and $\mu_i'\colon Q'\to\mathcal{L}_i$ are bijections such that both (μ_1,μ_2,μ_3) and (μ_1',μ_2',μ_3') are coordinatizations of the 3-net $(\mathcal{P};\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$, then the mappings $\alpha_i=(\mu_i')^{-1}\mu_i$, $1\le i\le 3$, are bijections $Q\to Q'$ that fulfil*

$$xy=z\ \Leftrightarrow\ \alpha_1(x)\alpha_2(y)=\alpha_3(z).$$

Proof. The mapping α_i is a bijection since both $\mu_i\colon Q\to\mathcal{L}_i$ and $\mu_i'\colon Q'\to\mathcal{L}_i$ are bijections, $i\in\{1,2,3\}$. Let $x,y,z\in Q$ be such that $xy=z$. Then $\mu_1(x)\cap\mu_2(y)\cap\mu_3(z)\neq\emptyset$, by the definition of coordinatization. This can be written as $\mu_1'\alpha_1(x)\cap\mu_2'\alpha_2(y)\cap\mu_3'\alpha_3(z)\neq\emptyset$ since $\mu_i'\alpha_i=\mu_i'(\mu_i')^{-1}\mu_i=\mu_i$. This means that $\alpha_1(x)\alpha_2(y)=\alpha_3(z)$ holds in Q_2 since (μ_1',μ_2',μ_3') is a coordinatization of $(\mathcal{P};\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$. □

Isotopy. Suppose that Q_1 and Q_2 are quasigroups. Suppose that α, β and γ are bijections $Q_1\to Q_2$. The triple (α,β,γ) is called an *isotopy* $Q_1\to Q_2$ if and only if

$$\forall x,y,z\in Q\colon\ xy=z\ \Leftrightarrow\ \alpha(x)\beta(y)=\gamma(z).$$

This can be also expressed as $\gamma(xy)=\alpha(x)\beta(y)$. The fact that α, β and γ are bijections means that is suffices to verify $xy=z\Rightarrow\alpha(x)\beta(y)=\gamma(z)$. Indeed, if $\alpha(x)\beta(y)=\gamma(z)$ and $xy=z'$, then $\alpha(x)\beta(y)=\gamma(z')$ and $z=z'$.

Quasigroups Q_1 and Q_2 are called *isotopic* if and only if there exists an isotopy $Q_1 \to Q_2$.

Theorem. *Quasigroups Q_1 and Q_2 are isotopic if and only if there exists a 3-net $(\mathcal{P}; \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ that may be coordinatized both by Q_1 and Q_2.*

Proof. By the Proposition any two quasigroups coordinatizing the same 3-net are isotopic. Suppose now that $(\alpha_1, \alpha_2, \alpha_3)$ is an isotopy $Q_1 \to Q_2$. We shall show that both Q_1 and Q_2 may be used to coordinatize the 3-net of Q_2 that consists of row lines r_b, column lines c_b and symbol lines s_b, $b \in Q_2$. A coordinatization (ν_1, ν_2, ν_3) by Q_2 is defined straightforwardly as $\nu_1(b) = r_b$, $\nu_2(b) = c_b$ and $\nu_3(b) = s_b$. The triple (ν_1, ν_2, ν_3) coordinatizes the 3-net since $xy = z$ if and only if $r_x \cap c_y \cap s_z \neq \emptyset$, for any $x, y, z \in Q_2$.

A coordinatization $(\lambda_1, \lambda_2, \lambda_3)$ by Q_1 is defined so that $\lambda_1(a) = r_{\alpha_1(a)}$, $\lambda_2(a) = c_{\alpha_2(a)}$ and $\lambda_3(a) = s_{\alpha_3(a)}$, for each $a \in Q_1$. Suppose that $x, y, z \in Q_1$. By the definition, $\lambda_1(x) \cap \lambda_2(y) \cap \lambda_3(z)$ is equal to $r_{\alpha_1(x)} \cap c_{\alpha_2(y)} \cap s_{\alpha_3(z)}$. This is nonempty if and only if $\alpha_1(x) \cdot \alpha_2(y) = \alpha_3(z)$. Since $(\alpha_1, \alpha_2, \alpha_3)$ is an isotopy $Q_1 \to Q_2$, the latter equality holds if and only if $xy = z$. Therefore $xy = z$ if and only if $\lambda_1(x) \cap \lambda_2(y) \cap \lambda_3(z) \neq \emptyset$. This verifies that $(\lambda_1, \lambda_2, \lambda_3)$ is a coordinatization of the 3-net upon $Q_2 \times Q_2$. □

Elementary algebraic properties of isotopies. *Suppose that $(\alpha, \beta, \gamma) \colon Q_1 \to Q_2$ and $(\delta, \epsilon, \eta) \colon Q_2 \to Q_3$ are isotopies. Then both $(\delta\alpha, \epsilon\beta, \eta\gamma) \colon Q_1 \to Q_3$ and $(\alpha^{-1}, \beta^{-1}, \gamma^{-1}) \colon Q_2 \to Q_1$ are isotopies.*

To verify the former property consider $x, y \in Q_1$. Then $\delta\alpha(x) \cdot \epsilon\beta(y) = \eta(\alpha(x) \cdot \beta(y)) = \eta\gamma(xy)$. To verify the latter property consider $x', y' \in Q_2$. There exist unique $x, y \in Q_1$ such that $x' = \alpha(x)$ and $y' = \beta(y)$. Now, $\alpha^{-1}(x')\beta^{-1}(y') = xy = \gamma^{-1}\gamma(xy) = \gamma^{-1}(\alpha(x)\beta(y)) = \gamma^{-1}(x'y')$.

Note that $\alpha \colon Q_1 \to Q_2$ is an *isomorphism* if and only if (α, α, α) is an isotopomism $Q_1 \to Q_2$.

Autotopies and the left nucleus. Let Q be a quasigroup. An isotopy $Q \to Q$ is called an *autotopy*. All autotopies form a group. This group will be denoted by $\mathrm{Atp}(Q)$.

Consider $a \in Q$ and recall that L_a denotes the left translation of the element a. The triple $(L_a, \mathrm{id}_Q, L_a)$ is an isotopy if and only if $L_a(x) \cdot \mathrm{id}_Q(y) = L_a(xy)$ for all $x, y \in Q$. This is the same as

$$a \cdot xy = ax \cdot y \text{ for all } x, y \in Q.$$

All $a \in Q$ that fulfil this conditions form a subset of Q that is called the *left nucleus.* It is denoted by $N_\lambda(Q)$. Elements of $N_\lambda(Q)$ are those elements of Q that may be described by saying that they 'associate upon the left'.

Exercise. Let G be a group. Describe $\mathrm{Atp}(G)$.